Homework 3 (Sol)

Math 622

February 25, 2016

1. (i) Q is a Levy process so

$$m = \mathbb{E}(Q(1)) = b1\lambda_1 + b_2\lambda_2.$$

(ii) Consider the price model

$$dS(t) = \alpha S(t)dt + S(t-)dM(t), S(0) = 1.$$

$$S(t) = \exp\Big((\alpha - m)t + \log(1 + b_1)N_1(t) + \log(1 + b_2)N_2(t)\Big).$$

So $K = 1, a_0 = \alpha - m, a_i = \log(1 + b_i), i = 1, 2.$
(iii) We have $V(t) = c(t, S(t))$ where

$$\begin{aligned} c(t,x) &:= e^{-r(T-t)}\mathbb{E}\Big[\Big(xe^{a_0(T-t)+a_1[N_1(T)-N_1(t)]+a_2[N_2(T)-N_2(t)]} - K\Big)^+\Big] \\ &= e^{-r(T-t)}\sum_{i,j=0}^{\infty}\Big(xe^{a_0(T-t)+a_1 i+a_2 j} - K\Big)^+ e^{-(\lambda_1+\lambda_2)(T-t)}\frac{\lambda_1^i\lambda_2^j(T-t)^{i+j}}{i!j!}, \end{aligned}$$

where the second equality comes from the independence of $N_1(T) - N_1(t), N_2(T) - N_2(t)$ so that their joint distribution is the product of their individual distributions.

(iv) We have

$$dS(t) = rS(t)dt + S(t-)d[Q(t) - (m + r - \alpha)t],$$

so we want $Q(t)$ to be a Levy process under $\mathbb{Q}$ and

$$\mathbb{E}^{\mathbb{Q}}(Q(1)) = m + r - \alpha.$$

From problem 3 of homework 2, we can choose $\mathbb{Q}$ such that N_i are Poisson processes under $\mathbb{Q}$ with rates $\tilde{\lambda}_i, i = 1, 2$. Thus the requirement about $Q(t)$ being Levy is fulfilled. Moreover we want

$$\mathbb{E}^{\mathbb{Q}}(Q(1)) = \sum_{i=1}^{2} b_i \mathbb{E}^{\mathbb{Q}}(N_i(1)) = \sum_{i=1}^{2} b_i \tilde{\lambda}_i = m + r - \alpha.$$

This sets up an equation for $\tilde{\lambda}_i, i = 1, 2$. If we can find $\tilde{\lambda}_i > 0$ such that the equation is satisfied, then we can define the change of measure in the manner similar to 3(ii).

(v) Recalling that $m = b_1\lambda_1 + b_2\lambda_2$, we can re-write the above equation as

$$b_1\tilde{\lambda}_1 = b_1\lambda_1 + r - \alpha + b_2(\lambda_2 - \tilde{\lambda}_2).$$

This is to exploit the fact that $b_1 > 0 > b_2$. The idea is to choose, if possible, $\tilde{\lambda}_2 > 0$ so that the RHS > 0. Then since $b_1 > 0$ we can solve for $\tilde{\lambda}_1 > 0$. But this is indeed always possible since $b_2 < 0$, it is clear that $\lim_{x\to\infty} b_2(\lambda_2 - x) = \infty$ so we just have to choose $\tilde{\lambda}_2$ positive and large enough. It follows that indeed there are infinitely many values of such pairs $\tilde{\lambda}_1, \tilde{\lambda}_2$.

(vi) From a similar reasoning to part (v), we need to choose $\mathbb{Q}$ such that N_i are Poisson processes under $\mathbb{Q}$ with rates $\tilde{\lambda}_i, i = 1, 2$, where $\tilde{\lambda}_i$ satisfies

$$\begin{aligned} b_1\tilde{\lambda}_1 + b_2\widetilde{\lambda}_2 &= b_1\lambda_1 + b_2\lambda_2 + r - \alpha_1 \\ \widetilde{\lambda}_1 &= \frac{r - \alpha_2}{\sigma_2}. \end{aligned}$$

Then it is clear that

$$\begin{aligned} \widetilde{\lambda}_2 &= \frac{b_1\lambda_1 + b_2\lambda_2 + r - \alpha_1 - b_1\frac{r-\alpha_2}{\sigma_2}}{b_2} \\ \widetilde{\lambda}_1 &= \frac{r - \alpha_2}{\sigma_2}. \end{aligned}$$

And we require

$$\begin{aligned} r - \alpha_2 &> 0 \\ b_1\lambda_1 + b_2\lambda_2 + r - \alpha_1 - b_2\frac{r - \alpha_2}{\sigma_2} &> 0 \ . \end{aligned}$$

If these conditions are satisfied, then the risk neutral measure $\mathbb{Q}$ is unique, since by the above solution, $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ are unique.

2.

a) Explicitly identify a constant θ and a compound Poisson process $\bar{Q}$ such that

$$S(t) = S(0)\exp\{\sigma W(t) + \theta t + \bar{Q}(t)\}.$$

At the ith jump of Q:

$$1 + \Delta Q(t_i) = 1 + e^{Z_i} - 1 = e^{Z_i}.$$

Thus

$$\prod_{i=1}^{N(t)} (1 + \Delta Q(t_i)) = e^{\sum_{i=1}^{N(t)} Z_i} := e^{\bar{Q}(t)},$$

where $\bar{Q}(t)$ is compound Poisson with jump distribution Z_i and rate λ. Thus

$$\begin{aligned} S(t) &= S(0)\exp\Big[(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big] \prod_{i=1}^{N(t)} (1 + \Delta Q(t_i)) \\ &= \exp\Big[(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t) + \bar{Q}(t)\Big], \end{aligned}$$

and $\theta = \alpha - \frac{1}{2}\sigma^2, \bar{Q}(t) = \sum_{i=1}^{N(t)} Z_i$.

b) Since

$$m := E(Q(1)) = \lambda E(e^{Z_i} - 1) = \lambda(e^{\frac{1}{2}} - 1),$$

and

$$dS(t) = rS(t)dt + \sigma S(t)dWt + S(t-)[dQ(t) - (r - \alpha)dt],$$

it follows that we need $r - \alpha = m$ or $\alpha = r - m$ for the model to be risk neutral.

c) We have

$$S(T) = S(t)\exp\Big[\theta(T - t) + \sigma(W(T) - W(t)) + \bar{Q}(T) - \bar{Q}(t)\Big].$$

Thus by the Independence lemma,

$$\begin{aligned} V(t) &= e^{-r(T-t)} E\Big[\Big(S(t)\exp\big[\theta(T - t) + \sigma(W(T) - W(t)) + \bar{Q}(T) - \bar{Q}(t)\big] - K\Big)^+ \big|\mathcal{F}(t)\Big] \\ &= c(t, S(t)), \end{aligned}$$

where

$$c(t,x) = e^{-r(T-t)} E\Big[H(x, Y(T-t))\Big],$$

and

$$\begin{aligned} H(x,y) &= (xe^y - K)^+; \\ Y(s) &= \sigma W(s) + \theta s + \bar{Q}(s); \\ \theta &= \alpha - \frac{1}{2}\sigma^2 = r - m - \frac{1}{2}\sigma^2, \end{aligned}$$

m is given in part b).

Note that here we use the fact that both Brownian motion $W(t)$ and compound Poisson $\bar{Q}(t)$ has stationary distribution. So $W(T) - W(t) = W(T-t)$ and $\bar{Q}(T) - \bar{Q}(t) = \bar{Q}(T-t)$ (in distribution).

d) Note that

$$\bar{Q}(T-t) = \sum_{i=1}^{N(T-t)} Z_i,$$

and $N(T-t)$ has Poisson $\lambda(T-t)$ distribution. Recall from problem 2, conditioning on $N(T-t) = n, \bar{Q}(T-t)$ has Normal $(0, n)$ distribution.

Also because $\bar{Q}(t)$ and $W(t)$ are independent $\bar{Q}(T-t) + \sigma W(T-t)$ has $N(0, n + \sigma^2(T-t)$ distribution.

Thus we see that conditioning on $N(T-t) = n$, we can write (as far as distribution is concerned)

$$(xe^{Y(T-t)} - K)^+ = (axe^{\nu_n U} - K)^+,$$

where U has standard normal distribution, $a = e^{\theta(T-t)}$, $\nu_n = \sqrt{n + \sigma^2(T-t)}$.

Thus if we let

$$\bar{\kappa}(\tau, x, \nu) := e^{-r\tau} E\Big[\big(xe^{\nu U} - K\big)^+\Big],$$

then by the Independence Lemma (Since $N(T-t)$ is independent of the rest of the random variables in the expression of $Y(T-t)$, also see problem 2c)

$$c(t,x) = \sum_{n=0}^{\infty} \bar{\kappa}(T-t, ax, \nu_n) \frac{(\lambda(T-t))^n}{n!} e^{-\lambda(T-t)}.$$

where

$$\begin{aligned} a &= e^{\theta(T-t)}; \\ \nu_n &= \sqrt{n+\sigma^2(T-t)}. \end{aligned}$$

3.
(a)
We have

$$m := E(Q(1)) = \lambda E(Y_1) = 2\frac{3}{20} = \frac{3}{10}.$$

thus

$$dS(t) = S(t-)d(Q(t) - mt),$$

so $S(t)$ is a martingale.
(b) Denote $m = \frac{3}{10}, a_1 = \frac{3}{4}, a_2 = \frac{-3}{4}, \lambda_1 = \frac{6}{5}, \lambda_2 = \frac{4}{5}$ then

$$Q(t) = a_1 N_1(t) + a_2 N_2(t),$$

where N_1, N_2 are indendent Poisson processes with rates λ_1, λ_2 and

$$S(t) = e^{-mt+\log(1+a_1)N_1(t)+\log(1+a_2)N_2(t)}.$$

So

$$S(T) = S(t)e^{-m(T-t)+\log(1+a_1)[N_1(T)-N_1(t)]+\log(1+a_2)[N_2(T)-N_2(t)]}.$$

Thus by the Independence Lemma,

$$E\Big[(K - S(T))^+ \big| \mathcal{F}(t)\Big] = c(t, S(t)),$$

where

$$\begin{aligned} c(t,x) &= E\Big[(xe^{-m(T-t)+\log(1+a_1)N_1(T-t)+\log(1+a_2)N_2(T-t)} - K)^+\Big] \\ &= \sum_{i,j=1}^{\infty}(xe^{-m(T-t)+\log(1+a_1)i+\log(1+a_2)j} - K)^+ e^{-(\lambda_1+\lambda_2)(T-t)}\frac{\lambda_1^i\lambda_2^j(T-t)^{i+j}}{i!j!}. \end{aligned}$$

4. a) From the dynamics of $S(t)$:

$$\Delta S(t) =: S(t) - S(t-) = \sqrt{S(t-)}S(t-)\Delta Q(t).$$

Thus

$$S(t) = S(t-)[1 + \sqrt{S(t-)}\Delta Q(t)].$$

So it is clear that

$$S(t) = \begin{cases} S(t-), & \text{if } \triangle Q(t) = 0; \\ S(t-)\Big(1 + \sqrt{S(t-)}\Big), & \text{if } \triangle N_1(t) = 1; \\ S(t-)\Big(1 - (1/2)\sqrt{S(t-)}\Big), & \text{if } \triangle N_2(t) = 1. \end{cases}$$

Also since

$$\begin{aligned} \Delta c(t, S(t)) &= c(t, S(t)) - c(t, S(t-)) \\ &= c(t, S(t-)[1 + \sqrt{S(t-)}\Delta Q(t)]) - c(t, S(t-)), \end{aligned}$$

coupled with the fact that N_1, N_2 do not jump at the same time, we have

$$\begin{aligned} \triangle c(t, S(t)) &= \Big[c\big(t, S(t-)(1 + \sqrt{S(t-)}\big) - c(t, S(t-))\Big]\triangle N_1(t) \\ &\quad + \Big[c\big(t, S(t-)(1 - \sqrt{S(t-)}/2)\big) - c(t, S(t-))\Big]\triangle N_2(t) \end{aligned}$$

b) Apply Ito's formula, noting the fact that we need to compensate $-1dt$ for both processes N_1, N_2 to make them martingales, the equation for $c(t, x)$ is

$$\begin{aligned} -rc(t, x) &+ \frac{\partial}{\partial t}c(t, x) + (r - \frac{1}{2}\sqrt{x})x\frac{\partial}{\partial x}c(t, x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}c(t, x)x^2 \\ &+ [c(t, x(1 + \sqrt{x})) - c(t, x)] + [c(t, x(1 - \frac{\sqrt{x}}{2})) - c(t, x)] = 0, 0 \le t < T, x > 0; \\ c(T, x) &= (x - K)^+, x > 0. \end{aligned}$$

6. Let

$$\begin{aligned} dS(t) &= \alpha(t)S_t d_t + \sigma S(t)dW(t) + S(t-)dQ(t), \\ S(0) &= 1. \end{aligned}$$

where $Q(t) = \sum_{i=1}^{N}(t)Y_i$ is a compound Poisson process,

$$\begin{aligned} Y_i &= \frac{1}{4} \text{ with probability } p \\ &= -\frac{1}{3} \text{ with probability } 1 - p. \end{aligned}$$

We require $p > 1 - p$ so that the positive jump arrives faster on average than the negative jumps. This implies $p > \frac{1}{2}$.

Let $U_1, U_2, \ldots$ be independent, identically distributed random variables satisfying $\mathbf{P}(U_i > 0) = 1$ and $E[U_i] = 1/4$. These represent the random jolts causing positive price jumps. Let $N_1(t)$ denote the process counting the positive jumps. Assume it is a Poisson process with rate λ_1, independent of $U_1, U_2, \ldots$.

Let $V_1, V_2, \ldots$ be independent, identically distributed random variables with $\mathbf{P}(-1 < V_i < 0) = 1$ and $E[V_i] = -1/3$. These represent the jolts causing negative price jumps. We assume they arrive as a Poisson process with rate λ_2, independent of all random variables and processes previously defined. By criterion (iv) of the model specification, one should assume $\lambda_1 > \lambda_2$. Let

$$Q(t) = \sum_{j=1}^{N_1(t)} U_i + \sum_{k=1}^{N_2(t)} V_i.$$

This is a compound Poisson process with mean $E[Q(t)] = \lambda_1/4 - \lambda_2/3$.

Let α denote the instantaneous mean rate of return, and assume it is constant and deterministic. Criterion (i) says that between jumps $dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$ where W is a Brownian motion. In keeping with the general form of a Lévy process, assume that W is independent of N_1, N_2, $U_1, U_2, \ldots$ and $V_1, V_2, \ldots$.

The model we propose is

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t) + S(t-)\,d[Q(t) - (\lambda_1/4 - \lambda_2/3)t].$$