Homework 3 (Full version)

Math 622

February 25, 2016

1. On a probability space $(\Omega, \mathbb{P})$ let N_1, N_2 be independent Poisson processes with rate λ_1, λ_2 and $\mathcal{F}(t)$ a filtration for N_1, N_2. Assume $b_1 > 0 > b_2 > -1$ and let

$$Q(t) := b_1 N_1(t) + b_2 N_2(t).$$

(i) Find m so that $M(t) = Q(t) - mt$ is a $\mathcal{F}(t)$-martingale.

(ii) Consider the price model

$$dS(t) = \alpha S(t)dt + S(t-)dM(t), S(0) = 1.$$

Write down a solution in the form $S(t) = Ke^{a_0 t + a_1 N_1(t) + a_2 N_2(t)}$; identify the constants a_0, a_1, a_2 and K.

(iii) Let $\alpha = r$, where r is the risk free interest rate, for the model in part (ii), then the measure $\mathbb{P}$ is risk-neutral. So the price of a Euro-call option that pays $V(T) = (S(T) - K)^+$ at time T is

$$V(t) = e^{-r(T-t)}\mathbb{E}[(S(T) - K)^+ | \mathcal{F}(t)].$$

Find an explicit formular for $V(t)$ in the style of the formula (11.7.3) on page 507 of Shreve. Your final answer will be a doubly infinite sum.

(iv) Suppose now that $a \neq r$ for the model in part (ii). Show how we can define a risk-neutral measure $\mathbb{Q}$ for the model in (ii) (Hint: use the result in 3 (ii)).

(v) Show that there are in fact many different risk-neutral measures for the setting in (iv).

(vi) Suppose that now we consider a market with 2 assets:

$$dS_1(t) = \alpha_1 S_1(t)dt + S_1(t-)dM(t), S_1(0) = 1$$
$$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t-)dN_1(t), S_2(0) = 1,$$

where $\sigma_2 > 0$. A risk neural probability $\mathbb{Q}$ for this market must be such that $e^{-rt}S_1(t)$ and $e^{-rt}S_2(t)$ are $\mathcal{F}(t)$ martingales under $\mathbb{Q}$. Show how we can define a risk neutral measure $\mathbb{Q}$ for this market. What conditions must $\alpha_1, \alpha_2, \sigma_1, \sigma_2, \lambda_1, \lambda_2, r$ satisfy for this measure change to be valid? When is $\mathbb{Q}$ unique? (It is helpful to look at the discussion in Shreve's pagge 516).

2. *Merton's jump diffusion process.* Let $Z_1, Z_2, \dots$ be independent standard normal random variables that are independent of W and N, where W and N are independent, W is a Brownian motion, and N is a Poisson process with rate λ. Let $Q(t) = \sum_1^{N(t)} [e^{Z_i} - 1]$. Consider the price model

$$dS(t) = \alpha S(t)\, dt + S(t)\sigma\, dW(t) + S(t-) dQ(t),$$

This price process is called Merton's jump diffusion.

a) Explicitly identify a constant θ and a compound Poisson process $\bar{Q}$ such that

$$S(t) = S(0) \exp\{\sigma W(t) + \theta t + \bar{Q}(t)\}.$$

b) Let r be the risk-free interest rate. What must α be so that this model is risk-neutral?

c) Let α be chosen as in b), so that the model is risk-neutral. If we were to use this risk-neutral model to price a call option at strike K and expiry T, we would find the price is $V(t) = e^{-r(T-t)} E\Big[\big(S(T) - K\big)^+ \big| \mathcal{F}(t)\Big]$. Show that $V(t) = c(t, S(t))$, where $c(t, x)$ is given in the form

$$c(t, x) = e^{-r(T-t)} E\Big[H(x, Y(T-t))\Big],$$

where $Y(s)$ has the form $Y(s) = \sigma W(s) + \theta s + \bar{Q}(s)$. Your answer should explicitly define H and should express θ in terms of r, λ, and σ.

d) Let

$$\bar{\kappa}(\tau, x, \nu) := e^{-r\tau} E\Big[\big(x e^{\nu U} - K\big)^+\Big],$$

where U is a standard normal random variable. By conditioning on $N(T-t)$ in the style of Exercise 2, find constants a and $\nu_1, \nu_2, \dots$ so that

$$c(t, x) = \sum_{n=0}^{\infty} \bar{\kappa}(T-t, ax, \nu_n) \frac{(\lambda(T-t))^n}{n!} e^{-\lambda(T-t)}.$$

(The constants a and $\nu_1, \nu_2, \dots$ will depend on $T-t$ and other parameters of the model.)

Remark. This is an interesting formula because $\bar{\kappa}$ has the following explicit form, closely related to the Black-Scholes formula:

$$\bar{\kappa}(\tau, x, \nu) = e^{-r\tau}\Big[xe^{\nu^2/2}N\big(\frac{\nu^2 - \ln(K/x)}{\nu}\big) - KN\big(-\nu^{-1}\ln(K/x)\big)\Big].$$

3. Let $Q(t)$ be the compound Poisson process

$$Q(t) = \sum_{k=1}^{N(t)} Y_i,$$

where $Y_1, Y_2, \ldots$ are i.i.d. with $\mathbf{P}\Big(Y_i = \frac{3}{4}\Big) = \frac{3}{5}$ and $\mathbf{P}\Big(Y_i = -\frac{3}{4}\Big) = \frac{2}{5}$, and where N is a Poisson process with rate 2. Let $N_1(t)$ count the number of jumps of Q by $3/4$ and let $N_2(t)$ count the number of jumps of Q by $-3/4$.

Consider,

$$dS(t) = -(3/10)S(t)\,dt + S(t-)\,dQ(t), \qquad S(0) = 1. \tag{1}$$

(a) If S solves equation (1), is it a martingale or not? Explain briefly.

(b) There is a function $c(t,x)$ such that $E\Big[(K - S(T))^+ \big| \mathcal{F}(t)\Big] = c(t, S(t))$. Find an explicit expression for $c(t,x)$ as a doubly infinite sum.

4. Consider the risk neutral model,

$$dS(t) = rS(t)\,dt + S(t)\,d\tilde{W}(t) + \sqrt{S(t-)}S(t-)\,d[Q(t) - t/2]$$

Here $Q(t) = N_1(t) - (1/2)N_2(t)$ is a compound Poisson process , where both N_1 and N_2 are independent Poisson processes, both with rate $\lambda = 1$. As usual, N_1 and N_2 are independent of W. In this model the jumps the price experiences in its returns are affected by the level of the price. There is no longer an explicit solution since the factor $\sqrt{S(t-)}$ appears in the last term.

This price equation has a solution which is a Markov process, and so it will be possible to write the price of a call option in the form

$$V(t) = e^{-r(T-t)}\tilde{E}\Big[(S(T) - K)^+ \big| \mathcal{F}(t)\Big] = c(t, S(t))$$

The purpose of this problem is to derive an equation for c of the type found in Theorem 11.7.7 for the model (11.7.27).

a) Show that

$$S(t) = \begin{cases} S(t-), & \text{if } \triangle Q(t) = 0; \\ S(t-)\Big(1 + \sqrt{S(t-)}\Big), & \text{if } \triangle N_1(t) = 1; \\ S(t-)\Big(1 - (1/2)\sqrt{S(t-)}\Big), & \text{if } \triangle N_2(t) = 1. \end{cases}$$

and thus that

$$\begin{aligned} \triangle c(t, S(t)) &= \Big[c\big(t, S(t-)(1 + \sqrt{S(t-)}\big) - c(t, S(t-))\Big]\triangle N_1(t) \\ &\quad + \Big[c\big(t, S(t-)(1 - \sqrt{S(t-)}/2)\big) - c(t, S(t-))\Big]\triangle N_2(t) \end{aligned}$$

b) By applying Itô's rule for jump processes to $e^{-rt}c(t, S(t))$ and insisting that it be a martingale, find an equation that, approximately, is of the type derived in Theorem 11.7.7. It will look very similar, except that the terms $c(t, x(1 + y_m))$ in equation (11.7.33) will be modified.

5. Create a price model for a single asset price with the following properties:

(i) Normally the price follows a Black-Scholes type of evolution with a constant volatility.

(ii) Occasionally the price jumps up by an amount that is, on average, one quarter of the pre-jump price. These jumps arrive according to a Poisson process.

(iii) Occasionally the price jumps down by an amount that is, on average, one third of the pre-jump price. These price jolts arrive according to a Poisson process independently of the positive jumps.

(iv) The positive jumps arrive at a faster rate than the negative jumps.

There is no one right answer to this problem. Your model will contain different parameters. Try to leave as many as possible as free constants—you would want to be able to choose these parameters to fit empirical data if you were to implement the model. However, the conditions (i)—(iv) might imply relation(s) among the parameters and you should specify these. *You may want to utilize this fact in the construction of the model: the sum of 2 compound Poisson processes is still a coumpound Poisson process in the following sense: if $N_1(t), N_2(t)$ are indendent Poisson processeswith* *rates* λ_1, λ_2, X_i *i.i.d and* Y_i *i.i.d random variables such that* $\{X_i\}$ *is independent of* $\{Y_i\}$ *then*

$$Q(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{j=1}^{N_2(t)} Y_j$$

is also a compound Poisson process. Can you see why? Can you write $Q(t)$ *in the regular form of a Compound Poisson process?*