Homework 2 (Due 02/17/2016)

Math 622

February 11, 2016

1. Let N be a Poisson process with rate λ and filtration $\mathcal{F}(t)$. Define

$$Y(t) := \exp\Big(uN(t) - \lambda t(e^u - 1)\Big),$$

that is Y is the exponential martingale associated with N. Use stochastic calculus (Ito's formula) for jump processes to show that

$$Y(t) = 1 + \int_0^t (e^u - 1)Y(s-)dM(s),$$

where $M(t) = N(t) - \lambda t$ and conclude that $Y(t)$ is a martingale w.r.t $\mathcal{F}(t)$.

2. Let $N_1(t), N_2(t)$ be independent Poisson processes with rate λ_1, λ_2 and $\mathcal{F}(t)$ a filtration for both N_1, N_2. Also define $M_i(t) = N_i(t) - \lambda_i t, i = 1, 2$.

(i) Show that the probability that N_1 and N_2 have the same jump time is 0 (Hint: Apply two dimensional Ito's formula for processes with jumps to $M_1(t)M_2(t)$ and take expectations on both sides).

(ii) Let

$$Y(t) = \exp\big(u_1 N_1(t) + u_2 N_2(t) - \lambda_1 t(e^{u_1} - 1) - \lambda_2 t(e^{u_2} - 1)\big).$$

Use a similar technique like problem 1 (i.e do not use direct computation) to show that $Y(t)$ is a martingale with respect to $\mathcal{F}(t)$. (Part (i) may also be helpful here).

4. Let $N(t)$ be a Poisson(λ) process and W_t a Brownian motion. Show that N_t, W_t are independent.

5. *Some probability theory and an application to compound Poisson processes.* Let Z be a random variable. Then by the averaging property of conditional expectation,

$E[U] = E\big[E[U\big|Z]\big]$. (For example, see equation (2.3.17) in Shreve with $A = \Omega$ and $\mathcal{G} = \sigma(Z)$.)

Let $Q(t) = \sum_{j=1}^{N(t)} Y_i$ be a compound Poisson process, so that N is a Poisson process with rate λ and $Y_1, Y_2, \ldots$ is a sequence of independent, identically distributed random variables that are independent of N.

a) Consider evaluating $E[G(Q(t))]$ for some function G. For every integer $n \geq 0$, let $\ell(n) := E\Big[G\big(\sum_{j=1}^{n} Y_i\big)\Big]$ (when $n = 0$, interpret this as $\ell(0) = G(0)$). By conditioning on $N(t)$—that is, let $N(t)$ play the role of Z above—show that

$$E[G(Q(t))] = \sum_{n=0}^{\infty} \ell(n) \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

b) Assume that $Y_1, Y_2 \ldots$ are independent, normal random variables with mean μ and variance σ^2. Explain why the condition distribution of $Q(t)$ given $N(t) = n$ is normal with mean $n\mu$ and variance $n\sigma^2$. Using the technique of part a), show that,

$$E[Q^2(t)] = (\sigma^2 + \mu^2)\lambda t + (\mu\lambda t)^2,$$

and that,

$$E\big[e^{uQ(t)}\big] = e^{\lambda t(e^{u\mu+\sigma^2 u^2/2} - 1)}.$$

c) Prove as a general principle. Assume that Y is independent of $X_1, \ldots, X_M$. Show that $E\big[H(Y, X_1, \ldots, X_M)\big] = E\big[h(X_1, \ldots, X_M)\big]$, where $h(x_1, \ldots, x_M) = E\big[H(Y, x_1, \ldots, x_M)\big]$.

6. Let $T_i, i = 1, ..., k$ be independent exponentially distributed random variables with rate $\lambda_i, i = 1, ..., k$.

(i) Let $U = \min_{i=1,\ldots,k} T_i$ and $V = \max_{i=1,\ldots,k} T_i$. Find the density functions of U and V.

(ii) Show that $P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.

7.

(i) Suppose S_t satisfies

$$S(t) = 1 + \int_0^t \alpha S(u) du + \sum_{0<u\leq t} \Delta J(u),$$

where $J(u)$ is a pure jump function. Solve for an explicit formula for $S(t)$.

Explanation: So far we've studied the model of

$$\begin{aligned} S(t) &= 1 + \int_0^t \alpha S(u)du + \int_0^t S(u-)dJ(u) \\ &= 1 + \int_0^t \alpha S(u)du + \sum_{0<u\leq t} S(u-)\Delta J(u). \end{aligned}$$

It is natural to ask how the solution changes if the term $S(u-)$ disappears in the equation. There are 2 ways to solve this question: a) let $0 < t_1 < t_2 < ...$ be the jump times of J. Solve for $S(t)$ on each interval $t_i < t < t_{i+1}$ (note the strict inequality) and consider what happens at each t_i. b) Note that we have a simpler way to write $\sum_{0<u\leq t} \Delta J(u)$. Apply Ito's formula to $e^{-\alpha t}S_t$ and see what happens.

(ii) Now suppose S_t satisfies

$$S(t) = 1 + \int_0^t \alpha(u)S(u)du + \sum_{0<u\leq t} \sigma \Delta J(u).$$

Solve for an explicit formula for $S(t)$.