Homework 2 (Sol)

Math 622

February 19, 2016

1. Apply Ito's formula:

$$\begin{aligned}
Y(t) &= 1+\int_0^t -\lambda(e^u-1)Y(s)ds + \sum_{0<s\le t} e^{uN(s)-\lambda s(e^u-1)} - e^{uN(s-)-\lambda s(e^u-1)} \\
&= 1+\int_0^t -\lambda(e^u-1)Y(s)ds + \sum_{0<s\le t} e^{uN(s-)-\lambda s(e^u-1)}(e^{u\Delta N(s)}-1) \\
&= 1+\int_0^t -\lambda(e^u-1)Y(s)ds + \sum_{0<s\le t} Y(s-)(e^u-1)\Delta N(s) \\
&\quad \text{because } \Delta N(s)=1 \\
&= 1+\int_0^t (e^u-1)Y(s-)dM(s).
\end{aligned}$$

2. (i) We have

$$\begin{aligned}
M_1(t)M_2(t) &= \int_0^t M_1(s-)dM_2(s) + \int_0^t M_2(s-)dM_1(s) \\
&\quad + \sum_{0<s\le t}[M_1(s)-M_1(s-)][M_2(s)-M_2(s-)].
\end{aligned}$$

Note that $\mathbb{E}(M_i(t)) = M_i(0) = 0, i = 1,2$ and by their independence, as well as the stochastic integrals are martingales with expectations 0:

$$\mathbb{E}\Big(\sum_{0<s\le t}[M_1(s)-M_1(s-)][M_2(s)-M_2(s-)]\Big) = 0.$$

Since $\sum_{0<s\le t}[M_1(s)-M_1(s-)][M_2(s)-M_2(s-)] \ge 0$ with probability 1 (the jumps of M_i come from N_i and N_i only jumps up), it must be that with probability 1,

$$\sum_{0<s\le t}[M_1(s)-M_1(s-)][M_2(s)-M_2(s-)] = 0,$$

hence with probability 1,

$$[M_1(s) - M_1(s-)][M_2(s) - M_2(s-)] = 0.$$

But this means that with probability 1, M_1, M_2 cannot have the same jump points (otherwise at the common jump point, it has to be that $[M_1(s) - M_1(s-)][M_2(s) - M_2(s-)] > 0$).

(ii) Let

$$Y(t) = \exp\big(u_1 N_1(t) + u_2 N_2(t) - \lambda_1 t(e^{u_1} - 1) - \lambda_2 t(e^{u_2} - 1)\big).$$

Denote for $i = 1, 2$

$$\begin{aligned} Z_i(t) &:= \exp\Big(u_i N_i(t) - \lambda_i t(e^{u_i} - 1)\Big) \\ M_i(t) &:= N_i(t) - \lambda_i(t). \end{aligned}$$

Then by problem 1,

$$Z_i(t) = 1 + \int_0^t (e^{u_i} - 1) Z_i(s-) dM_i(s).$$

Note that Z_i jumps exactly at the jump points of M_i, which are exactly the jump points of N_i. So by part (i), with probability 1

$$\sum_{0<s\le t} [Z_1(s) - Z_1(s-)][Z_2(s) - Z_2(s-)] = 0.$$

Thus by Ito's formula,

$$\begin{aligned} Y(t) &= Z_1(t)Z_2(t) = \int_0^t Z_1(s-)dZ_2(s) + \int_0^t Z_2(s-)dZ_1(s) \\ &\quad + \sum_{0<s\le t} [Z_1(s) - Z_1(s-)][Z_2(s) - Z_2(s-)] \\ &= \int_0^t Z_1(s-)(e^{u_2} - 1)Z_2(s-)dM_2(s) + \int_0^t Z_2(s-)(e^{u_1} - 1)Z_1(s-)dM_1(s). \end{aligned}$$

Thus $Y(t)$ is a martingale.

3.

(i) We want to show

$$Y(t) = \exp\big(u_1 N_1(t) + u_2 N_2(t) - \lambda_1 e^{a_1} t(e^{u_1} - 1) - \lambda_2 e^{a_2} t(e^{u_2} - 1)\big)$$

is a $\mathbb{Q}$ - martingale.

This is equivalent to show that a.

$$Z(t) := \exp\Big[a_1 N_1(t) + a_2 N_2(t) - \lambda_1 t(e^{a_1} - 1) - \lambda_2 t(e^{a_2} - 1)\Big]$$

is a $\mathbb{P}$-martingale.

b. $YZ(t)$ is a $\mathbb{P}$-martingale.

Part a. is clear from 2(ii). For part b. note that

$$YZ(t) = \exp\Big[(a_1 + u_1)N_1(t) + (a_2 + u_2)N_2(t) - \lambda_1 t(e^{a_1+u_1} - 1) - \lambda_2 t(e^{a_2+u_2} - 1)\Big].$$

So $YZ(t)$ being a martingale also follows from 2(ii).

(ii) We want for $i = 1, 2$

$$\tilde{\lambda}_i = \lambda_i e^{a_i}.$$

Clearly then

$$a_i = \log\left(\frac{\tilde{\lambda}_i}{\lambda_i}\right).$$

Thus

$$\begin{aligned} Z(T) &= \exp\Big[\log\left(\frac{\tilde{\lambda}_1}{\lambda_1}\right) N_1(T) + \log\left(\frac{\tilde{\lambda}_2}{\lambda_2}\right) N_2(T) - \lambda_1 T(\frac{\tilde{\lambda}_1}{\lambda_1} - 1) - \lambda_2 T(\frac{\tilde{\lambda}_2}{\lambda_2} - 1)\Big] \\ &= \prod_{i=1}^{2} \left(\frac{\tilde{\lambda}_i}{\lambda_i}\right)^{N_i(T)} e^{(\lambda_i - \tilde{\lambda}_i)T}. \end{aligned}$$

4. See Shreve.

5. a)

$$\begin{aligned} E[G(Q(t))] &= E[G(\sum_{i=1}^{N(t)} Y_i)] \\ &= E\Big[E\big[G(\sum_{i=1}^{N(t)} Y_i)\Big| N(t)\big]\Big] \end{aligned}$$

By the Independence lemma,

$$E\big[G(\sum_{i=1}^{N(t)} Y_i)\Big| N(t)\big] = l(N(t))$$

where

$$l(n) := \big[G(\sum_{i=1}^{n} Y_i)\big].$$

Thus

$$\begin{aligned} E[G(Q(t))] &= E\Big[E\big[G(\sum_{i=1}^{N(t)} Y_i)\Big| N(t)\big]\Big] \\ &= E(l(N(t))), \end{aligned}$$

and the final result follows.

b) Since $Q(t) = \sum_{i=1}^{N(t)} Y_i$, if $N(t) = n$ then $Q(t) = \sum_{i=1}^{n} Y_i$. It follows from probability theory that sum of independent normal is normal. So $Q(t)$ has normal distribution with

$$\begin{aligned} E(Q(t)) &= \sum_{i=1}^{n} E(Y_i) = n\mu \\ Var(Q(t)) &= \sum_{i=1}^{n} Var(Y_i) = n\sigma^2. \end{aligned}$$

Apply part a with

$$l(n) \;:=\; E((\sum_{i=1}^{n} Y_i)^2) = n\sigma^2 + n^2\mu^2,$$

it is clear that

$$\begin{aligned} E[G(Q(t)] &= E(N(t)\sigma^2 + N(t)^2\mu^2) \\ &= \lambda t\sigma^2 + [\lambda t + (\lambda t)^2]\mu^2 \\ &= \lambda t(\sigma^2 + \mu^2) + (\mu\lambda t)^2. \end{aligned}$$

Apply part a with

$$l(n) :=: E(e^{u\sum_{i=1}^{n} Y_i}) = e^{nu\mu + \frac{1}{2}nu^2\sigma^2}.$$

We have

$$\begin{aligned} E[G(Q(t)] &= E(e^{N(t)(u\mu + \frac{1}{2}u^2\sigma^2)}) \\ &= e^{\lambda t(e^{u\mu + \frac{1}{2}u^2\sigma^2} - 1)}. \end{aligned}$$

c) Let $\mathcal{F}$ be the sigma algebra generated by $X_i, i = 1, ..., M$. Then by the Independence Lemma,

$$E[H(Y, X_1, X_2, ..., X_M)] = E(E(H(Y, X_1, ..., X_M)|\mathcal{F})).$$

Now by the independence lemma,

$$E(H(Y, X_1, ..., X_M)|\mathcal{F}) = h(X_1, ..., X_M)$$

where $h(x_1, ..., x_M) := E(H(Y, x_1, ..., x_M))$. Thus

$$E[H(Y, X_1, X_2, ..., X_M)] = E(E(H(Y, X_1, ..., X_M)|\mathcal{F})) = E(h(X_1, ..., X_M)).$$

6. (i)

$$\begin{aligned} \mathbb{P}(U \geq t) &= \mathbb{P}(T_i \geq t, \forall i) = \prod_{i=1}^{k} \mathbb{P}(T_i \geq t) \\ &= e^{-\sum_{i=1}^{k} \lambda_i t}. \end{aligned}$$

Thus

$$f_U(t) = \sum_{i=1}^{k} \lambda_i e^{-\sum_{i=1}^{k} \lambda_i t}.$$

Note that U has distribution Exponential $(\sum_{i=1}^{k} \lambda_i)$.

$$\begin{aligned} \mathbb{P}(V \leq t) &= \mathbb{P}(T_i \leq t, \forall i) = \prod_{i=1}^{k} \mathbb{P}(T_i \leq t) \\ &= \prod_{i=1}^{k} (1 - e^{-\lambda_i t}). \end{aligned}$$

Thus

$$f_V(t) = \sum_{i=1}^{k} \left[\lambda_i e^{-\lambda_i t} \prod_{j \neq i} (1 - e^{-\lambda_j t}) \right].$$

(ii)

$$
\begin{aligned}
P(T_1 < T_2) &= \int_0^\infty P(T_1 < t | T_2 = t) f_{T_2}(t) dt \\
&= \int_0^\infty P(T_1 < t) f_{T_2}(t) dt \\
&= (1 - e^{-\lambda_1 t}) \lambda_2 e^{-\lambda_2 t} dt \\
&= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}
$$

7.

(i) Clearly

$$
S(t) = 1 + \int_0^t \alpha S(u) du + J(t).
$$

Apply Ito's formula

$$
\begin{aligned}
e^{-\alpha t} S(t) &= 1 + \int_0^t -\alpha e^{-\alpha u} S(u) du + \int_0^t e^{-\alpha u} dS^c(u) + \sum_{0<u\le t} e^{-\alpha u} S(u) - e^{-\alpha u} S(u-) \\
&= 1 + \int_0^t -\alpha e^{-\alpha u} S(u) du + \int_0^t e^{-\alpha u} \alpha S(u) du + \sum_{0<u\le t} e^{-\alpha u} S(u) - e^{-\alpha u} S(u-) \\
&= 1 + \sum_{0<u\le t} e^{-\alpha u} \Delta J(u)
\end{aligned}
$$

Thus

$$
S(t) = e^{\alpha t} + \sum_{0<u\le t} e^{\alpha(t-u)} \Delta J(u)
$$

(ii) Apply Ito's formula to $e^{-\int_0^t \alpha(u)du} S(t)$, after cancellation we have

$$
e^{-\int_0^t \alpha(u)du} S(t) = 1 + \sum_{0<u\le t} e^{-\int_0^u \alpha(s)ds} \sigma \Delta J(u).
$$

Thus

$$
S(t) = e^{\int_0^t \alpha(u)du} + \sum_{0<u\le t} e^{\int_u^t \alpha(s)ds} \sigma \Delta J(u)
$$