Homework 11 (Sol)

Math 622

May 4, 2016

1.

The solution follows the method in class for deriving the Black-Scholes-Merton formula with random interest rate (see Theorem 9.4.2 in Shreve).

According to Theorem 9.2 (here the role of $\nu(t)$ is played by $3-t$),

$$\widetilde{W}^{T_3}(t) = \widetilde{W}(t) - \int_0^t (3-s)\,ds = \widetilde{W}(t) + (1/2)[(3-t)^2 - 9]$$

is a Brownian motion under $\widetilde{\mathbf{P}}^{(3)}$, for $t \leq 3$, and

$$d_t\text{For}_S(t) = \text{For}_S(t)[\sigma - (3-t)]\,d\widetilde{W}^{T_3}(t) = \text{For}_S(t)[t+\sigma-3]\,d\widetilde{W}^{T_3}(t).$$

Then $\text{For}_S(3) = \dfrac{s_0}{B(0,3)} e^{X-\nu^2/2}$, where

$$X = \int_0^3 [s+\sigma-3]\,d\widetilde{W}^{T_3}(s),$$

is, under the measure $\widetilde{\mathbf{P}}^{(3)}$, a Gaussian random variable with mean 0 and variance $\nu^2 = \int_0^3 [s+\sigma-3]^2\,ds = (1/3)[\sigma^3 - (\sigma+3)^3]$. Therefore, using Theorem 2 in Lecture Notes 12, the forward price of the call at $t=0$ is

$$\begin{aligned}
\frac{V(0)}{B(0,3)} &= \tilde{E}^{T_3}\Big[(S(3)-K)^+\Big] \\
&= \frac{s_0}{B(0,3)} N\left(\frac{\ln(s_0/B(0,3)K) + \nu^2/2}{\nu}\right) - KN\left(\frac{\ln(s_0/B(0,3)K) - \nu^2/2}{\nu}\right)
\end{aligned}$$

As a result,

$$V(0) = s_0 N\left(\frac{\ln(s_0/B(0,3)K) + \nu^2/2}{\nu}\right) - KB(0,3)N\left(\frac{\ln(s_0/B(0,3)K) - \nu^2/2}{\nu}\right).$$

2. (Shreve, 10.9). By equation (5) in Lecture Notes 12,

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sum_{j=1}^d \sigma_j(u,T)\,dW_j(u). \tag{1}$$

Following the derivation of section 10.3.2, show that

$$d_t\Big(- \int_t^T f(t,v)\,dv\Big) = R(t)\,dt - \alpha^*(t,T)\,dt - \sum_{j=1}^d \sigma_j^*(t,T)\,dW_j(t)$$

By applying Itô's rule to $B(t,T) = e^{-\int_t^T f(t,v)\,dv}$, it follows that

$$dB(t,T) = B(t,T)\Big[R(t) - \alpha^*(t,T) + \frac{1}{2}\sum_{j=1}^d (\sigma_j^*(t,T))^2\Big]\,dt$$
$$- B(t,T)\sum_{j=1}^d \sigma_j^*(t,T)\,dW_j(t)$$

Write

$$\begin{aligned} dB(t,T) &= B(t,T)\Big[R(t) - \alpha^*(t,T) + \frac{1}{2}\sum_{j=1}^d (\sigma_j^*(t,T))^2 + \sum_{j=1} \sigma_j^*(t,T)\Theta_j(t)\Big]\,dt \\ &\quad - B(t,T)\sum_{j=1}^d \sigma_j^*(t,T)\,d\widetilde{W}_j(t) \end{aligned}$$

where $\widetilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(s)\,ds$. When we change measure to make $\widetilde{W}(t) = (\widetilde{W}_1(t), \ldots, \widetilde{W}_d(t))$ a Brownian motion, the model will be risk-neutral if

$$\alpha^*(t,T) = \frac{1}{2}\sum_{j=1}^d (\sigma_j^*(t,T))^2 + \sum_{j=1}^d \sigma_j^*(t,T)\Theta_j(t).$$

Take derivatives on both sides with respect to T to obtain,

$$\alpha(t,T) = \sum_{j=1}^d \sigma_j^*(t,T)\sigma_j(t,T) + \sum_{j=1}^d \sigma_j(t,T)\Theta_j(t).$$

(ii) Suppose there is a solution $\Theta(t) = (\Theta_1(t), \ldots, \Theta_d(t))$. If $T_1, \ldots, T_d$ is a set of distinct times then

$$\alpha(t,T_i) = \sum_{j=1}^d \sigma_j^*(t,T_i)\sigma_j(t,T_i) + \sum_{j=1}^d \sigma_j(t,T_i)\Theta_j(t)$$

for $1 \le i \le d$. If the matrix $[\sigma_j(t,T_i)]_{1\le i,j\le d}$ is invertible, this system has a unique solution $\Theta(t)$ for all $t \le \min_k T_k$.

3. a) In this case $\sigma^*(t,T) = \int_t^T tv\,dv = (1/2)t(T^2 - t^2)$. In order that the model be arbitrage-free, there must exist a solution $\theta(t)$ to $\alpha(t,T) = \sigma(t,T)[\sigma^*(t,T) + \theta(t)]$, or

$$\frac{T^3t^2}{2} + 5Tt - \frac{Tt^4}{2} = tT[(1/2)t(T^2 - t^2) + \theta(t)].$$

Subtracting $\dfrac{T^3t^2}{2} - \dfrac{Tt^4}{2}$ from both sides leaves $5Tt = tT\theta(t)$. Therefore there is a solution with $\theta(t) \equiv 5$, and so the model is arbitrage-free.

b) Let $\sigma_1(t,T) = 1$ and $\sigma_2(t,T) = 2T$. Then $\sigma_1^*(t,T) = T - t$ and $\sigma_2^*(t,T) = \int_t^T 2udu = T^2 - t^2$.

From Exercise 10.9, for the model to be arbitrage-free there must be a solution $\theta_1(t), \theta_2(t)$, independent of T, to

$$\alpha(t,T) = T - t - 2Tt^2 = [T - t + \theta_1(t)] + 2T[T^2 - t^2 + \theta_2(t)].$$

By cancellation of terms common to both sides, $(\theta_1(t), \theta_2(t))$ must solve

$$0 = \theta_1(t) + 2T\theta_2(t) + 2T^3$$

for all $0 \le t \le T \le \bar{T}$. If this were true, then taking partial derivatives with respect to T on both sides implies $2\theta_2(t) + 6T^2 = 0$. But this contradicts the condition that $\theta_2(t)$ is independent of T, and hence there can be no solution of the required form. Therefore, we conclude that the given model is not arbitrage-free.

4. (Shreve, Exercise 10.11) The value at $t = 0$ of a payment of δK at T is $\delta K B(0,T)$. The value at $t = 0$ of a series of payments of δK at time $T_1, \ldots, T_{n+1}$ is thus $\delta K \sum_{j=1}^{n+1} B(0,T_j)$. By Theorem 10.4.1, the value at $t = 0$ of a payment of amount $\delta L(T_{j-1}, T_{j-1})$ at T_j is $\delta B(0,T_j)L(0,T_{j-1})$—see equation (10.4.5). The value of a contract at $t = 0$ promising fixed legs in return for paying floating legs is therefore

$$\delta K \sum_{j=1}^{n+1} B(0,T_j) - \delta \sum_{j=1}^{n+1} B(0,T_j)L(0,T_{j-1}) \tag{2}$$

5. For the one-factor Vasicek model, $dR(t) = (a - bR(t))\,dt + \sigma R(t)\,d\widetilde{W}(t)$, the results of section 10.3.5 show that

$$d[D(t)B(t,T)] = -\sigma^*(t,T)[D(t)B(t,T)]\,d\widetilde{W}(t),$$

where $\sigma^*(t,T) = \frac{\sigma}{b}\Big(1 - e^{-b(T-t)}\Big)$. By (10.4.9) and (10.4.15),

$$\begin{aligned} dL(t,T) &= \frac{1+\delta L(t,T)}{\delta L(t,T)}\Big[\sigma^*(t,T+\delta) - \sigma^*(t,T)\Big] L(t,T)\, d\widetilde{W}^{T+\delta}(t) \\ &= [1+\delta L(t,T)]\frac{\sigma e^{-b(T-t)}(1-e^{-b\delta})}{\delta b}\, d\widetilde{W}^{T+\delta}(t). \end{aligned}$$

Let $Y(t) = 1 + \delta L(t,T)$. Then it follows that

$$dY(t) = \delta dL(t,T) = Y(t)\beta(t,T)\, d\widetilde{W}^{T+\delta}(t),$$

where $\beta(t,T) = \frac{\sigma e^{-b(T-t)}(1-e^{-b\delta})}{b}$. This has the solution

$$Y(t) = [1+\delta L(0,T)]\exp\{\int_0^t \beta(u,T)\, d\widetilde{W}^{T+\delta}(u) - \frac{1}{2}\int_0^t \beta^2(u,T)\, du\}.$$

Let $B = \int_0^T \beta(u,T)\, d\widetilde{W}^{T+\delta}(u)$. This is a normal random variable with mean 0 and variance $\int_0^T \beta^2(u,T)\, du$. Then

$$L(T,T) = \delta^{-1}[Y(T)-1] = \delta^{-1}\Big[(1+\delta L(0,T))e^{B-(1/2)\int_0^T \beta^2(u,T)\, du} - 1\Big].$$

Let $V(0)$ denote the price of a caplet at strike K for $[T, T+\delta]$. The $T+\delta$-forward price is thus

$$\begin{aligned} \frac{V(0)}{B(0,T+\delta)} &= \tilde{E}^{T+\delta}\Big[(L(t,T)-K)^+\Big] \\ &= \tilde{E}^{T+\delta}\Bigg[\Big(\delta^{-1}(1+\delta L(0,T))e^{B-(1/2)\int_0^T \beta^2(u,T)\, du} - \delta^{-1} - K\Big)^+\Bigg] \end{aligned}$$

The Black-Scholes formula tells us how to price this. It is the same as the price of a call at strike $\delta^{-1} + K$, when $\sigma^2 T = \int_0^T \beta^2(u,T)\, du$, $r = 0$ and the initial price is $\delta^{-1} + L(0,T)$. This is

$$(\delta^{-1} + L(0,T))N(\bar{d}_+) - (\delta^{-1}+K)N(\bar{d}_-),$$

where

$$d_\pm = \frac{1}{\sqrt{\int_0^T \beta^2(u,T)\, du}}\Bigg[\log\frac{1+\delta L(0,T)}{1+\delta K} \pm \frac{1}{2}\int_0^T \beta^2(u,T)\, du\Bigg].$$

We could compute $\int_0^T \beta^2(u,T)\, du$ explicitly, but have not done so here. Finally,

$$V(0) = B(0,T+\delta)\Bigg[(\delta^{-1} + L(0,T))N(\bar{d}_+) - (\delta^{-1}+K)N(\bar{d}_-)\Bigg].$$