Homework 10 (Sol)

Math 622

April 28, 2016

2. *Shreve, 10.2.*

Let Y_1 and Y_2 solve (10.2.59)-(10.2.60). By the Markov property of (Y_1, Y_2), we know that there is a function $f(t, y_1, y_2)$ such that $B(t,T) = f(t, Y_1(t), Y_2(t))$. It must satsify $f(t, y_1, y_2) \equiv 1$. If it is twice continuously differentiable, Itô's rule implies

$$\begin{aligned}
&d[D(t)f(t,Y_1(t),Y_2(t))] \\
&= D(t)\Big\{ -R(t)f(t,Y_1(t),Y_2(t)) + f_t(t,Y_1(t),Y_2(t)) \\
&\quad + (\mu - \lambda_1 Y_1(t)) f_{y_1}(t,Y_1(t),Y_2(t)) - \lambda_2 Y_2(t) f_{y_2}(t,Y_1(t),Y_2(t)) \\
&\quad + \frac{1}{2} Y_1(t) f_{y_1y_1}(t,Y_1(t),Y_2(t)) + \sigma_{21} Y_1(t) f_{y_1y_2}(t,Y_1(t),Y_2(t)) \\
&\quad + \frac{1}{2}(\alpha + \beta Y_1(t) + \sigma_{21}^2 Y_1(t)) f_{y_2y_2}(t,Y_1(t),Y_2(t)) \Big\} dt \\
&\quad + D(t) f_{y_1}(t,Y_1(t),Y_2(t)) \sqrt{Y_1(t)}\, d\widetilde{W}_1(t) \\
&\quad + D(t) f_{y_2}(t,Y_1(t),Y_2(t)) \Big[\sigma_{21}\sqrt{Y_1(t)}\, d\widetilde{W}_1(t) + \sqrt{\alpha + (\beta + \sigma_{21}^2)Y_1(t)}\, d\widetilde{W}_2(t)\Big]
\end{aligned}$$

We know $D(t)B(t,T) = D(t)f(t, Y_1(t), y_2(t))$ must be a martingale; by assumption $R(t) = \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t)$. Assuming the stochastic integral terms in the above formula are martingales, $D(t)f(t, Y_1(t), Y_2(t))$ will be a martingale if

$$\begin{aligned}
&-(\delta_0 + \delta_1 y_1 + \delta_2 y_2) f + f_t + (\mu - \lambda_1 y_1) f_{y_1} - \lambda_2 y_2 f_{y_2} \\
&\quad + \frac{1}{2} y_1 f_{y_1y_1} + \sigma_{21} y_1 f_{y_1y_2} + \frac{1}{2}(\alpha + (\beta + \sigma_{21}^2) y_1) f_{y_2y_2} = 0
\end{aligned}$$

The boundary condition is $f(T, y_1, y_2) = 1$ for all y_1, y_2.

The idea is to look for a solution of the form $f(t, y_1, y_2) = e^{-y_1 C_1(T-t) - y_2 C_2(T-t) - A(T-t)}$, where $C_1(0) = C_2(0) = A(0) = 0$. Let $\tau = T - t$. Calculation yields $f_\tau = (y_1 C_1'(\tau) + y_2 C_2'(\tau) + A'(\tau)) f$, $f_{y_1} = -C_1(\tau)$, $f_{y_1y_1} = C_1^2(\tau) g$, $f_{y_2} = -C_2(\tau) f$,

$f_{y_2y_2} = C_2^2(\tau)f$, and $f_{y_1y_2} = C_1(\tau)C_2(\tau)f$. By substituting these into the previous equation and collecting terms,

$$\begin{aligned}0 &= f\Big\{y_1\Big[C_1' + \lambda_1 C_1 + \frac{1}{2}C_1^2 + \sigma_{21}C_1C_2 + \frac{1}{2}(\sigma_{21}^2 + \beta)C_2^2 - \delta_1\Big] \\ &\quad + y_2\left[C_2' + \lambda_2 C_2 - \delta_2\right] \\ &\quad + \Big[-\delta_0 + A' - \mu C_1 + \frac{1}{2}\alpha C_2^2\Big]\Big\}\end{aligned}$$

We can satisfy this equation by setting the coefficients of y_1 and y_2 equal to zero, as well as the final term:

$$\begin{aligned}C_1' &= -\lambda_1 C_1 - \frac{1}{2}C_1^2 - \sigma_{21}C_1C_2 - \frac{1}{2}(\sigma_{21}^2 + \beta)C_2^2 + \delta_1 \\ C_2' &= -\lambda_2 C_2 + \delta_2 \\ A' &= \mu C_1 - \frac{1}{2}\alpha C_2^2 + \delta = 0\end{aligned}$$

3. *Exercise 10.3* The derivation of a p.d.e. for $f(t, T, Y_1(t), Y_2(t)) = B(t, T)$ follows the derivation of (10.2.18) exactly, and leads again to this equation for f, except that now $\delta_0(t)$ is a function of t.

(i) Assume $f(t, T, y_1, y_2) = \exp\{-y_1C_1(t, T) - y_2C_2(t, T) - A(t, T)\}$ with $C_(T, T) = C_2(T, T) = A(T, T) = 0$, in order that $f(t, T, \cdot, \cdot) \equiv 1$. Then,

$$\begin{aligned}f_t(t, y_1, y_2) &= \Big[-y_1\frac{\partial C_1(t,T)}{\partial t} - y_2\frac{\partial C_2(t,T)}{\partial t} - \frac{\partial A(t,T)}{\partial t}\Big]f(t, y_1, y_2) \\ f_{y_1}(t, y_1, y_2) &= -C_1(t, T)f(t, y_1, y_2), \quad f_{y_2}(t, y_1, y_2) = -C_2(t, T)f(t, y_1, y_2) \\ f_{y_1y_1}(t, y_1, y_2) &= C_1^2(t, T)f(t, y_1, y_2), \quad f_{y_2y_2}(t, y_1, y_2) = C_2^2(t, T)f(t, y_1, y_2) \\ & f_{y_1y_2}(t, y_1, y_2) = C_1(, T)C_2(t, T)f(t, y_1, y_2)\end{aligned}$$

By plugging this into equation (10.2.18) (similar to the derivation of (10.2.22))

$$\begin{aligned}\Big[&\left(-\frac{\partial C_1}{\partial t} + \lambda_1 C_1 + \lambda_{21}C_2 - \delta_1\right)y_1 \\ &\quad + \left(-\frac{\partial C_2}{\partial t} + \lambda_2 C_2 - \delta_2\right)y_2 \\ &\quad\quad + \left(-\frac{\partial A}{\partial t} + \frac{1}{2}C_1^2 + \frac{1}{2}C_2^2 - \delta_0(t)\right)\Big]f = 0\end{aligned}$$

The coefficients of y_1, y_2 must be zero for this to be true for all y_1 and y_2, and hence the last term must be zero also. Thus

$$\frac{\partial C_1}{\partial t}(t, T) = \lambda_1 C_1(t, T) + \lambda_{21}C_2(t, T) - \delta_1$$

$$\begin{aligned}\frac{\partial C_2}{\partial t}(t,T) &= \lambda_2 C_2(t,T) - \delta_2 \\ \frac{\partial A}{\partial t}(t,T) &= \frac{1}{2}C_1^2(t,T) + \frac{1}{2}C_2^2(t,T) - \delta_0(t)\end{aligned}$$

(ii) If we look for solutions to the first two equations above in the form $C_1(t,T) = \bar{C}_1(T-t)$ and $C_2(t,T) = \bar{C}_2(T-t)$, then $\bar{C}_1$ and $\bar{C}_2$ satisfy (10.2.23) and (10.2.24) with the same initial condition, and hence their solutions are given as in (10.2.26) and (10.2.7). For example, $C_2(t,T) = (\delta_2/\lambda_2)(1-e^{-\lambda_2(T-t)})$.

(iii) By integrating the equation in part (i) for A in the variable t and noting $A(T,T)=0$,

$$-A(t,T) = \int_t^T \Big[\frac{1}{2}C_1^2(s,T) + \frac{1}{2}C_2^2(s,T) - \delta_0(s)\Big]\,ds.$$

(iv) Since $C_1(t,T) = \bar{C}_1(T-t)$, $\partial_T C_1(t,T) = -\partial_t C_1(t,T)$, and similarly for C_2. Using this and the result of (iii),

$$\begin{aligned}\frac{\partial A}{\partial T}(0,T) &= \int_0^T \Big[C_1(s,T)\frac{\partial C_1}{\partial T}(s,T) + C_2(s,T)\frac{\partial C_2}{\partial T}(s,T)\Big]\,ds \\ &\quad - (1/2)C^1(T,T) - (1/2)C_2(T,T) + \delta_0(T) \\ &= \int_0^T \Big[-C_1(s,T)\frac{\partial C_1}{\partial s}(s,T) - C_2(s,T)\frac{\partial C_2}{\partial s}(s,T)\Big]\,ds + \delta_0(T) \\ &= -\frac{1}{2}\int_0^T \frac{\partial}{\partial s}\Big[C_1^2(s,T) + C_2^2(s,T)\Big]\,ds + \delta(T) \\ &= \frac{1}{2}\Big[C_1^2(T,T) + C_2^2(t,T) - C_1^2(0,T) - C_2^2(0,T)\Big] + \delta_0(T) \\ &= \delta_0(T) - \frac{1}{2}\Big[C_1^2(0,T) + C_2^2(0,T)\Big]\end{aligned}$$

On the other hand, if $Y_1(0)$ and $Y_2(0)$ are the initial values of the factors in the Vasicek model,

$$\frac{\partial}{\partial T}\ln B(0,T) = -Y_1(0)\frac{\partial}{\partial T}C_1(0,T) - Y_2(0)\frac{\partial}{\partial T}C_2(t,T) - \frac{\partial}{\partial T}A(0,T).$$

Since $\frac{\partial}{\partial T}C_1(0,T) = -\frac{\partial}{\partial t}C_1(0,T) = -\lambda_1 C_1(0,T) - \lambda_{21}C_2(0,T) + \delta_1$ (see (ii)), and $\frac{\partial}{\partial T}C_2(0,T) = -\frac{\partial}{\partial t}C_2(0,T) = -\lambda_2 C_2(0,T) + \delta_1$ (see (ii)), we find by rearranging terms that

$$\frac{\partial}{\partial T}A(0,T) = Y_1(0)\Big[\lambda_1 C_1(0,T) + \lambda_{21}C_2(0,T) - \delta_1\Big] + Y_2(0)\Big[\lambda_2 C_2(0,T) - \delta_1\Big] - \frac{\partial}{\partial T}\ln B(0,T).$$

By comparing the two expressions for $\frac{\partial}{\partial T}A(0,T)$, it follows

$$\begin{aligned}\delta_0(T) &= Y_1(0)\Big[\lambda_1 C_1(0,T) + \lambda_{21}C_2(0,T) - \delta_1\Big] + Y_2(0)\Big[\lambda_2 C_2(0,T) - \delta_1\Big] - \frac{\partial}{\partial T}\ln B(0,T) \\ &\quad + \frac{1}{2}\Big[C_1^2(0,T) + C_2^2(0,T)\Big]\end{aligned}$$

Since there are explicit formulae for $C_1(0,T)$ and $C_2(0,T)$, this last equation yields an explicit formula for $\delta_0(T)$ in terms of the model parameters and $\frac{\partial}{\partial T}\ln B(0,T)$, which can be estimated from bond prices.

4. (Shreve, Exercise 10.7) (a) We have $B(t,T) = e^{-Y_1(t)C_1(T-t)-Y_2(t)C_2(T-t)-A(T-t)}$, where C_1, C_2, A solve (10.2.23)–(10.2.25). Recall that, then, $f(t,y_1,y_2) = e^{-y_1C_1(T-t)-y_2C_2(T-t)-A(T-t)}$ solves (10.2.18). Thus, by the discussion in Shreve leading up to (10.2.18),

$$\begin{aligned}d[D(t)B(t,T)] &= D(t)f_{y_1}(t,Y_1(t),Y_2(t))\,d\widetilde{W}_1(t) + D(t)f_{y_2}(t,Y_1(t),Y_2(t))\,d\widetilde{W}_2(t) \\ &= D(t)\Big[-C_1(T-t)f(t,Y_1(t),Y_2(t))\,d\widetilde{W}_1(t) - C_2(T-t)f(t,Y_1(t),Y_2(t))\,d\widetilde{W}_2(t)\Big] \\ &= -D(t)B(t,T)\Big(C_1(T-t),C_2(T-t)\Big)\cdot\Big(d\widetilde{W}_1(t),d\widetilde{W}_2(t)\Big)\end{aligned}$$

We deduce from (9.2.5) in Chapter 9—see also page 393—that

$$\Big(\widetilde{W}_1^T(t),\widetilde{W}_2^T(t)\Big) = \Big(\widetilde{W}_1(t) + \int_0^t C_1(T-u)\,du, \widetilde{W}_2(t) + \int_0^t C_2(T-u)\,du\Big)$$

(ii) The T-forward price of the call option is

$$\begin{aligned}V(0) &= B(0,T)V^{(T)}(0,T) = B(0,T)\tilde{E}^T\Big[\frac{(B(T,\bar{T})-K)^+}{B(T,T)}\Big] \\ &= B(0,T)\tilde{E}^T\Big[(B(T,\bar{T})-K)^+\Big] \\ &= B(0,T)\tilde{E}^T\Big[\Big(e^{-Y_1(T)C_1(\bar{T}-T)-Y_2(T)C_2(\bar{T}-T)-A(\bar{T}-T)} - K\Big)^+\Big]\end{aligned}$$

(iii) Let $X = -Y_1(T)C_1(\bar{T}-T) - Y_2(T)C_2(\bar{T}-T) - A(\bar{T}-T)$. We know that

$$d\begin{pmatrix} Y_1(t) \\ Y_2(t)\end{pmatrix} = \begin{bmatrix} -\lambda_1 & 0 \\ -\lambda_{21} & -\lambda_2\end{bmatrix}\begin{pmatrix} Y_1(t) \\ Y_2(t)\end{pmatrix}dt + \begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t)\end{pmatrix}$$

We can rewrite this in terms of $\widetilde{W}_1^T$ and $\widetilde{W}_2^T$ using the result of part (i):

$$d\begin{pmatrix} Y_1(t) \\ Y_2(t)\end{pmatrix} = \begin{bmatrix} -\lambda_1 & 0 \\ -\lambda_{21} & -\lambda_2\end{bmatrix}\begin{pmatrix} Y_1(t) \\ Y_2(t)\end{pmatrix}dt - \begin{pmatrix} C_1(T-t) \\ C_2(T-t)\end{pmatrix}dt + \begin{pmatrix} d\widetilde{W}_1^T(t) \\ d\widetilde{W}_2^T(t)\end{pmatrix}$$

This is a linear system of stochastic differential equations for $Y_1(t)$ and $Y_2(t)$. Under $\widetilde{\mathbf{P}}^T$, $(\widetilde{W}_1^T(t), \widetilde{W}_2^T(t))$ is a Brownian motion for $t \leq T$. We know that the solution to this equation is a Gaussian process if the initial conditions are deterministic, and so $(Y_1(t), Y_2(t))$ is jointly normal under $\widetilde{\mathbf{P}}^T$. Since linear combinations of jointly normal random variables is normal, X is a normal random variable under $\widetilde{\mathbf{P}}^T$.

(iv) The random variable X of part (iii) can be written $X = -\sigma Z + (\mu - (1/2)\sigma^2)$, where Z is standard normal and μ is chosen so that $\mu - \sigma^2/2$ is the means of X. The general formula behind the Black-Scholes formula is given in Theorem 2 of the class *Notes to Lecture 12*. Because

$$V(0) = B(0,T)\tilde{E}^T\Big[\Big(e^{-X} - K\Big)^+\Big].$$

where, as we have shown, X is normal under $\widetilde{\mathbf{P}}^T$, we are precisely in the context of this Theorem. By applying it

$$\begin{aligned} V(0) &= B(0,T)\Big[e^{\mu} N\left(\frac{\ln(e^{\mu}/K) + \sigma^2/2}{\sigma}\right) - KN\left(\frac{\ln(e^{\mu}/K) - \sigma^2/2}{\sigma}\right)\Big] \\ &= B(0,T)\Big[e^{\mu} N\left(\frac{\mu - \ln(K) + \sigma^2/2}{\sigma}\right) - KN\left(\frac{\mu - \ln(K) - \sigma^2/2}{\sigma}\right)\Big]. \end{aligned}$$