Homework 1 (Due 02/10/2016)

Math 622

February 5, 2016

1. Let $0 < a < b$. Let G be a càdlàg function of bounded variation. In the following, the notation $\int H(s)dG(s)$ will mean $\int_{(0,\infty)} H(s)dG(s)$ as in the lecture Note 1.

(i) Use the definition in Section 8.3.B Lecture 1 note to show that $\int \mathbf{1}_{(a,b]}dG(s) = G(b) - G(a)$.

(ii) Show that $\lim_{n\to\infty} \mathbf{1}_{(a,b+\frac{1}{n}]}(t) = \mathbf{1}_{(a,b]}(t)$ and $\lim_{n\to\infty} \mathbf{1}_{(a+\frac{1}{n},b]}(t) = \mathbf{1}_{(a,b]}(t)$.

(iii) Show that

$$\lim_{n\to\infty} \int \mathbf{1}_{(a,b+\frac{1}{n}]}(s)dG(s) = \lim_{n\to\infty} \int \mathbf{1}_{(a+\frac{1}{n},b]}(s)dG(s) = \int \mathbf{1}_{(a,b]}(s)dG(s).$$

(iv) Is it true that

$$\lim_{n\to\infty} \int \mathbf{1}_{(a,b-\frac{1}{n}]}(s)dG(s) = \int \mathbf{1}_{(a,b]}(s)dG(s)?$$

(iv) Is it true that

$$\lim_{n\to\infty} \int \mathbf{1}_{(a-\frac{1}{n},b]}(s)dG(s) = \int \mathbf{1}_{(a,b]}(s)dG(s)?$$

(v) Evaluate $\int \mathbf{1}_{(a,b)}(s)dG(s)$, $\int \mathbf{1}_{[a,b)}(s)dG(s)$, $\int \mathbf{1}_{[a,b]}(s)dG(s)$ (Hint: Approximate these integrands with left continuous functions, and use the Dominated Convergence Theorem) .

2. Let

$$G(t) = \begin{cases} 2t & , \quad 0 \le t < 1; \\ t^2 - 3 & , \quad 1 \le t < 2; \\ t + 1 & , \quad 2 \le t. \end{cases}$$

Evaluate $\int_0^3 s dG(s)$.

3. Let $0 < t_1 < t_2$ and $a_1, a_2 \in \mathbb{R}$. Define

$$G(t) = \begin{cases} 0 & , \quad 0 \le t < t_1; \\ a_1 & , \quad t_1 \le t < t_2; \\ a_1 + a_2 & , \quad t_2 \le t. \end{cases}$$

(i) Let $\sigma > 0$. Solve for $Z(t)$, where $Z(t)$ satisfies

$$Z(t) = 1 + \int_0^t \sigma Z(s-) dG(s).$$

(ii) Now let $\sigma(s)$ be a function of s. Solve for $Z(t)$, where $Z(t)$ satisfies

$$Z(t) = 1 + \int_0^t Z(s) ds + \int_0^t \sigma(s) Z(s-) dG(s).$$

4. (i) Let $X(t)$ be a Levy process and $\mathcal{F}(t)$ be a filtration for $X(t)$. Let $\mu t = \mathbb{E}(X(t))$ and $\sigma^2 t = Var(X(t))$. Show that $(X(t) - \mu t)^2 - \sigma^2 t$ is a martingale w.r.t. $\mathcal{F}(t)$.

(ii) Let $N(t)$ be a Poisson process and $\mathcal{F}(t)$ be a filtration for $N(t)$. Show $\exp\big(iuN(t) - \lambda t(e^{iu} - 1)\big)$ is a martingale w.r.t. $\mathcal{F}(t)$.

(iii) Show that the Geometric Poisson process discussed in Example 9.1 of Lecture note 1 is a martingale (w.r.t its own filtration), *without using Shreve's Theorem 11.4.5.*

5. Let $X(t)$ be a Levy process and $\mathcal{F}(t)$ a filtration for $X(t)$. Use Lemma 2.3.4 and Definition 2.3.6 in Shreve to show that $X(t)$ is a Markov process.

6. (i) Let J be a counting process, that is $J(0) = 0$, J has finitely many jumps on any finite intervals and $\Delta J(t) = 1$ at any jump point of J. Show that

$$\begin{aligned} \int_0^t J(u) dJ(u) &= \frac{J(t)(J(t)+1)}{2} \\ \int_0^t J(u-) dJ(u) &= \frac{J(t)(J(t)-1)}{2}. \end{aligned}$$

Let $N(t)$ be a Poisson process with rate λ and $\mathcal{F}(t)$ a filtration for $N(t)$.

(ii) Find an explicit formula for

$$X(t) := \int_0^t (N(s) - N(s-)) d(N(s) - \lambda s),$$

and conclude that $X(t)$ is not a martingale (w.r.t $\mathcal{F}(t)$). (Hint: Using the fact that if $f(t) = 0$ at all but finitely many points t, then $f(s-) = 0$ so that $\int_0^t f(s)ds = \int_0^t f(s-)ds = 0$, it should be almost immediate to guess what $X(t)$ is).

(iii) Show that

$$Y(t) := \int_0^t N(s-)d(N(s) - \lambda s),$$

is a martingale (w.r.t $\mathcal{F}(t)$).

Hint: Recall that $\int_0^t N(s-)d(N(s) - \lambda s) = \int_0^t N(s-)dN(s) - \int_0^t \lambda N(s-)ds$ and part (i) of this problem. You can also use the fact that

$$\mathbb{E}(\int_0^t N(u)du|\mathcal{F}(s)) = \int_0^s N(u)du + \int_s^t \mathbb{E}(N(u)|\mathcal{F}(s))du.$$

(iv) Show that

$$Z(t) := \int_0^t N(s)d(N(s) - \lambda s)$$

is not a martingale w.r.t $\mathcal{F}(t)$.

7. Extra credit (5pts).

Let $f(t)$ be defined on $[0, \infty)$. Fix $T > 0$. The total variation of f on $[0, T]$, denoted as $TV_f(T)$ is defined as the smallest (finite) number such that for all partitions $0 = t_0 < t_1 < t_2 < ... < t_n = T$

$$\sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| \le TV_f(T).$$

If there is no such number, we define $TV_f(T) = \infty$.

We also say f is a function of bounded variation (on $[0, \infty)$) if $TV_f(T) < \infty$ for all $T > 0$.

(i) Let A be an increasing function on $[0, \infty)$. Show that for all $T > 0$, $TV_A(T) = A(T) - A(0)$. Thus any increasing function is of bounded variation.

(ii) Let A_1, A_2 be increasing functions on $[0, \infty)$. Show that $TV_{A_1 - A_2}(T) \le TV_{A_1}(T) + TV_{A_2}(T)$. Thus the difference between two increasing functions is of bounded variation. This is the reason for definition 4.1 in Lecture note 1.

(iii) Let $G(t)$ be a function of bounded variation. Show that for any partition $0 = t_0 < t_1 < t_2 < ... < t_n = T$,

$$\sum_{i=0}^{n-1} (G(t_{i+1}) - G(t_i))^2 \le \max_i |G(t_{i+1}) - G(t_i)| TV_G(T).$$

(iv) We say a function f is uniformly continuous on $[0, T]$ if there exists a non-negative function $\rho, \lim_{t\to 0} \rho(t) = 0 = \rho(0)$ and for all $0 \le t, s < T$, $|f(t) - f(s)| \le \rho(|t-s|)$. Use the fact that a continuous function on $[0, T]$ is uniformly continuous to show that if G is continuous, G is of bounded variation then its quadratic variation $[G, G](T) = 0$ for any $T > 0$ (See Sheve's Definition 3.4.1)

(v) Show that the sample paths of Brownian motion is not of bounded variation with probability 1.