Solution to Homework 1

Math 622

February 11, 2016

1.

(i) Since $\mathbf{1}_{(0,b]}(t) = \mathbf{1}_{(0,a]}(t) + \mathbf{1}_{(a,b]}(t)$,

$$\begin{aligned}\int \mathbf{1}_{(a,b]}(s)dG(s) &= \int \mathbf{1}_{(0,b]}(s)dG(s) - \int \mathbf{1}_{(0,a]}(s)dG(s) \\ &= G(b) - G(0) - (G(a) - G(0)) = G(b) - G(a).\end{aligned}$$

(ii) $\mathbf{1}_{(a,b+\frac{1}{n}]}(t) - \mathbf{1}_{(a,b]}(t) = \mathbf{1}_{(b,b+\frac{1}{n}]}(t)$. Fix $t > 0$. If $t \leq b$ then or $t > b+1$ then $\mathbf{1}_{(b,b+\frac{1}{n}]}(t) = 0$ for all $n = 1, 2, ...$ If $t \in (b, b+1]$ then there is always N large enough such that $t > b + \frac{1}{N}$ or $\mathbf{1}_{(b,b+\frac{1}{N}]}(t) = 0$. That is for all t, $\mathbf{1}_{(b,b+\frac{1}{n}]}(t) \to 0$ as $n \to \infty$. So $\mathbf{1}_{(a,b+\frac{1}{n}]}(t) \to \mathbf{1}_{(a,b]}(t)$. The other statement can be showed similarly.

(iii)

$$\int \mathbf{1}_{(a,b+\frac{1}{n}]}(s)dG(s) = G(b + \frac{1}{n}) - G(a).$$

Since G is right continuous, $G(b + \frac{1}{n}) - G(a) \to G(b) - G(a) = \int \mathbf{1}_{(a,b]}(s)dG(s)$ as $n \to \infty$.

$$\int \mathbf{1}_{(a+\frac{1}{n},b]}(s)dG(s) = G(b) - G(a + \frac{1}{n}).$$

Again, since G is right continuous, $G(b)-G(a+\frac{1}{n}) \to G(b)-G(a) = \int \mathbf{1}_{(a,b]}(s)dG(s)$ as $n \to \infty$.

(iv)

$$\int \mathbf{1}_{(a,b-\frac{1}{n}]}(s)dG(s) = G(b - \frac{1}{n}) - G(a).$$

Since G is not necessarily left continuous at b, we only can conclude that $G(b - \frac{1}{n})-G(a) \to G(b-)-G(a)$, which is not $G(b)-G(a) = \int \mathbf{1}_{(a,b]}dG(s)$. So the statement may not be true.

Similarly,

$$\int \mathbf{1}_{(a-\frac{1}{n},b]}(s)dG(s) = G(b) - G(a-\frac{1}{n}) \to G(b) - G(a-),$$

which may not be the same as $G(b) - G(a)$.

(v) We have $\mathbf{1}_{(a,b)}(t) = \lim_{n\to\infty} \mathbf{1}_{(a,b-\frac{1}{n}]}(t)$. It is also clear that $|\mathbf{1}_{(a,b-\frac{1}{n}]}(t)| \le 1$. So by the Dominated Convergence Theorem,

$$\int \mathbf{1}_{(a,b)}(s)dG(s) = \lim_{n\to\infty} \int \mathbf{1}_{(a,b-\frac{1}{n}]}(s)dG(s) = \lim_{n\to\infty} G(b-\frac{1}{n}) - G(a) = G(b-) - G(a).$$

Similarly, since $\mathbf{1}_{[a,b)}(s) = \lim_{n\to\infty} \mathbf{1}_{(a-\frac{1}{n},b-\frac{1}{n}]}(s)$, $\int \mathbf{1}_{[a,b)}(s)dG(s) = G(b-) - G(a-)$. Since $\mathbf{1}_{[a,b]}(s) = \lim_{n\to\infty} \mathbf{1}_{(a-\frac{1}{n},b+\frac{1}{n}]}(s)$, $\int \mathbf{1}_{[a,b)}(s)dG(s) = G(b) - G(a-)$.

2.

$$\begin{aligned} \int_0^3 sdG(s) &= \int_0^1 s(2ds) + 1(G(1) - G(1-)) + \int_1^2 s(2sds) + 2(G(2) - G(2-)) + \int_2^3 sds \\ &= 1 + (-2-2) + \frac{2}{3}(8-1) + 2(3-1) + \frac{5}{2} = \frac{49}{6}. \end{aligned}$$

3. (i) Observe that $Z(t)$ satisfies a similar equation to equation (3) in Section 9 of Lecture note 1 with $\alpha(s) = \mu(s) = \gamma(s) = 0$ and $J(t) = \sigma G(t)$. Also clearly $Z(0) = 1$. So we can use the formula provided for $S(t)$ (equation (4))to conclude that

$$Z(t) = \prod_{s\le t}(1 + \sigma\Delta G(s)).$$

More specifically,

$$Z(t) = \begin{cases} 1 & , \quad 0 \le t < t_1 \\ (1+\sigma a_1) & , \quad t_1 \le t < t_2 \\ (1+\sigma a_1)(1+\sigma a_2) & , \quad t_2 \le t. \end{cases}$$

Alternatively, we can observe that $Z(t)$ is also a pure jump function, $Z(t)$ jumps at the same point as $G(t)$ (that is t_1, t_2) and the jump size of $Z(t)$ at these points is $\Delta Z(t) = Z(t-)\sigma\Delta G(t)$. For example, $Z(0) = 1$. At t_1, we get $Z(t_1) - Z(t_1-) = Z(t_1-)\sigma a_1$, and $Z(t_1-) = 1$, since Z jumps at t_1. Using this we arrive at the same answer for Z.

(ii) Also using equation (4) we get

$$Z(t) = e^t \prod_{s\le t}(1 + \sigma(s)\Delta G(s)).$$

Thus,

$$Z(t) = \begin{cases} e^t & , \quad 0 \le t < t_1 \\ e^t(1+\sigma(t_1)a_1) & , \quad t_1 \le t < t_2 \\ e^t(1+\sigma(t_1)a_1)(1+\sigma(t_2)a_2) & , \quad t_2 \le t. \end{cases}$$

4.(i) Note that

$$\begin{aligned} (X(t)-\mu t)^2 &= (X(t)-X(s)+X(s)-\mu t)^2 \\ &= (X(t)-X(s))^2 + 2(X(t)-X(s))(X(s)-\mu t) + (X(s)-\mu t)^2. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}[(X(t)-\mu t)^2|\mathcal{F}(t)] &= \sigma^2(t-s) + [\mu(t-s)]^2 + 2\mu(t-s)(X(s)-\mu t) + (X(s)-\mu t)^2 \\ &= \sigma^2(t-s) + (X(s)-\mu t+\mu(t-s))^2 = \sigma^2(t-s) + (X(s)-\mu s)^2. \end{aligned}$$

The result follows.

(ii) Here we need to use the fact that if X is Poisson(λ) then the characteristic function of X is

$$\mathbb{E}(e^{iuX}) = \exp(\lambda(e^{iu}-1)).$$

See for example Wikipedia on Characteristic functions. Then

$$\begin{aligned} \mathbb{E}(e^{iuN(t)}|F(s)) &= e^{iuN(s)}\mathbb{E}(e^{iu(N(t)-N(s))}|F(s)) \\ &= \exp(iuN(s))\exp(\lambda(t-s)(e^{iu}-1)). \end{aligned}$$

The result follows.

(iii) Observe that $(1+\sigma)^{N(t)} = \exp\big(N(t)\log(1+\sigma)\big)$. Hence

$$\begin{aligned} \mathbb{E}((1+\sigma)^{N(t)}|\mathcal{F}(s)) &= \exp(N(s)\log(1+\sigma))\mathbb{E}\Big[\exp\big((N(t)-N(s))\log(1+\sigma)\big)|\mathcal{F}(s)\Big] \\ &= (1+\sigma)^{N(s)}\exp\big(\lambda(t-s)(e^{\log(1+\sigma)}-1)\big) \\ &= (1+\sigma)^{N(s)}\exp\big(\sigma\lambda(t-s)\big). \end{aligned}$$

where we used the result for the characteristic function of Poisson above with $u = -i\log(1+\sigma))$. The result follows.

5. Let f be a Borel measurable function. For $0 < s < t$ and x a real number, define

$$g(x) = \mathbb{E}[f(X(t)-X(s)+x)].$$

Then by Lemma 2.3.6 in Shreve we have

$$\begin{aligned}\mathbb{E}[f(X(t))|\mathcal{F}(s)] &= \mathbb{E}[f(X(t)-X(s)+X(s))|\mathcal{F}(s)]\\ &= g(X(s)).\end{aligned}$$

Thus by definition 2.3.6, $X(t)$ is a Markov process.

6. (i) Observe the followings: $J(s)-J(s-)=1$ at any jump point of J ; for any t, the number of jumps of J on $[0,t]$ is $J(t)$; if s_i denotes the time of the ith jump of J then $J(s_i)=i$. Putting these together, we have

$$\begin{aligned}\int_0^t J(u)dJ(u) &= \sum_{s\le t} J(s)(J(s)-J(s-))\\ &= \sum_{i=1}^{J(t)} J(s_i) = \sum_{i=1}^{J(t)} i = \frac{J(t)(J(t)+1)}{2}.\end{aligned}$$

$$\begin{aligned}\int_0^t J(u-)dJ(u) &= \sum_{s\le t} J(s-)(J(s)-J(s-))\\ &= \sum_{i=1}^{J(t)} J(s_i-) = \sum_{i=1}^{J(t)}(i-1)\\ &= \sum_{i=1}^{J(t)-1} i = \frac{J(t)(J(t)-1)}{2}.\end{aligned}$$

(ii)

$$\begin{aligned}\int_0^t (N(s)-N(s-))d(N(s)-\lambda s) &= \int_0^t (N(s)-N(s-))dN(s)\\ &= \sum_{s\le t}(N(s)-N(s-))(N(s)-N(s-))\\ &= \sum_{s\le t} N(s)-N(s-) = N(t),\end{aligned}$$

where we used the fact that $N(s)-N(s-)=1$ so that $(N(s)-N(s-))(N(s)-N(s-))=N(s)-N(s-)$. Since $\mathbb{E}(N(t))=\lambda t$, not a constant, it is not a martingale.

(iii)

$$\begin{aligned}Y(t) &= \int_0^t N(s-)dN(s) - \lambda\int_0^t N(s)ds\\ &= \frac{N(t)(N(t)-1)}{2} - \lambda\int_0^t N(s)ds.\end{aligned}$$

Since $N(t)^2 = (N(t) - N(s) + N(s))^2 = (N(t) - N(s))^2 + 2(N(t) - N(s))N(s) + N^2(s)$,

$$\mathbb{E}(N(t)^2|\mathcal{F}(s)) = \lambda(t-s) + [\lambda(t-s)]^2 + 2\lambda(t-s)N(s) + N^2(s).$$

Also

$$\mathbb{E}(N(t)|\mathcal{F}(s)) = \lambda(t-s) + N(s).$$

So that

$$\mathbb{E}(\frac{N(t)(N(t)-1)}{2}|\mathcal{F}(s)) = \frac{1}{2}\big([\lambda(t-s)]^2 + N^2(s) - N(s)\big) + \lambda(t-s)N(s).$$

Now

$$\lambda\int_s^t \mathbb{E}(N(u)|\mathcal{F}(s))du = \lambda\int_s^t N(s) + \lambda(u-s)du = \lambda(t-s)N(s) + \frac{1}{2}[\lambda(t-s)]^2.$$

So putting all these calculations together we have

$$\begin{aligned}\mathbb{E}(Y(t)|\mathcal{F}(s)) &= \frac{1}{2}(N^2(s) - N(s)) + \lambda\int_0^s N(u)du \\ &= \frac{N(s)(N(s)-1)}{2} + \lambda\int_0^s N(u)du = Y(s),\end{aligned}$$

so Y is a martingale.

(iv) Since $Z(t) - Y(t) = X(t)$, $Z(t)$ cannot be a martingale (otherwise this equation forces $X(t)$ to be a martingale, which we have showed is not the case).

7. (i) Since A is increasing, for $t_i < t_{i+1}$, $|A(t_{i+1}) - A(t_i)| = A(t_{i+1}) - A(t_i)$. Thus for all partitions $0 = t_0 < t_1 < ... < t_n = T$:

$$\sum_{i=0}^{n-1}|A(t_{i+1}) - A(t_i)| = A(T) - A(0).$$

It follows that $TV_A(T) = A(T) - A(0)$.

(ii) Since

$$\begin{aligned}|(A_1 - A_2)(t_{i+1}) - (A_1 - A_2)(t_i)| &= |A_1(t_{i+1}) - A_1(t_i) - (A_2(t_{i+1}) - A_2(t_i))| \\ &\leq |A_1(t_{i+1}) - A_1(t_i)| + |A_2(t_{i+1}) - A_2(t_i)|,\end{aligned}$$

we have

$$\begin{aligned}\sum_{i=0}^{n-1}|(A_1 - A_2)(t_{i+1}) - (A_1 - A_2)(t_i)| &\leq \sum_{i=0}^{n-1}|A_1(t_{i+1}) - A_1(t_i)| + \sum_{i=0}^{n-1}|A_2(t_{i+1}) - A_2(t_i)| \\ &\leq TV_{A_1}(T) + TV_{A_2}(T).\end{aligned}$$

Thus by definition of $TV_{A_1-A_2}(T)$, we have $TV_{A_1-A_2}(T) \le TV_{A_1}(T) + TV_{A_2}(T)$.

(iii)

$$
\begin{aligned}
\sum_{i=0}^{n-1}(G(t_{i+1})-G(t_i))^2 &= \sum_{i=0}^{n-1}|G(t_{i+1})-G(t_i)||G(t_{i+1})-G(t_i)| \\
&\le \sum_{i=0}^{n-1}\max_i(|G(t_{i+1})-G(t_i)|)|G(t_{i+1})-G(t_i)| \\
&= \max_i(|G(t_{i+1})-G(t_i)|)\sum_{i=0}^{n-1}|G(t_{i+1})-G(t_i)|.
\end{aligned}
$$

(iv) From part (iii)

$$
\begin{aligned}
\sum_{i=0}^{n-1}(G(t_{i+1})-G(t_i))^2 &\le \max_i(|G(t_{i+1})-G(t_i)|)\sum_{i=0}^{n-1}|G(t_{i+1})-G(t_i)| \\
&\le \rho(\max_i(|t_{i+1}-t_i|))TV_G(T).
\end{aligned}
$$

As $\max_i(|t_{i+1}-t_i|) \to 0$, since $TV_G(T) < \infty$, the RHS of the above inequality goes to 0. Thus the quadratic variatio of G is 0.

(v) Since $[W(t), W(t)] = t$ with probability 1, $W(t)$ cannot be of bounded variation.