Math 622
Spring 2015
Midterm 2 - Form A
04/08/2015

Name (Print): ______________________

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- The parts of the problems are **not necessarily connected.** If you cannot do one part, you can assume the result of that part, if needed, to do the next part.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. Let $\widetilde{P}$ be a risk neutral measure. Let S_t have the dynamics of geometric Brownian motion under $\widetilde{P}$:

$$dS_t = (0.5)S_t dt + \sqrt{2}S_t d\widetilde{W}_t.$$

Consider a perpetual American option on S_t with payoff function $g(S_t)$, where $g(x) = \big(K - \sqrt{x}\big)^+$ and K is a positive constant. Let V_t be the value of this American option at time t, assuming it has not been exercised.

(a) (5 points) By the Markov property of S_t, there is a bounded function $v(x)$ such that $v(S_t) = V_t$. State the linear complimentarity equations for $v(x)$. (You only need to state what they are; and do not have to justify).

The linear complimentarity equations are:

$$\begin{aligned} v(x) &\geq (K - \sqrt{x})^+ \\ -(0.5)v + v_x(0.5)x + v_{xx}x^2 &\leq 0 \\ -(0.5)v + v_x(0.5)x + v_{xx}x^2 &= 0 \text{ on } v(x) > (K - \sqrt{x})^+. \end{aligned}$$

(b) (10 points) Assuming that the continuation region has the form $C = \Big\{x > L^*\Big\}$ and exercise region has the form $E = \Big\{0 \leq x \leq L^*\Big\}$ where $L^* \leq K^2$. By using the smooth pasting principle, solve for L^* and the solution $v(x)$ of the above linear complimentarity equations. (You don't have to verify that $v(x)$ satisfy the linear complimentarity equations here. It's in the next part).

Ans:

The general solution to

$$-(0.5)v + v_x(0.5)x + v_{xx}x^2 = 0$$

is

$$v(x) = Ax^{-0.5} + Bx.$$

Since $v(x)$ is bounded for $x > 0$, $v(x) = Ax^{-0.5}$. So the form of the solution is

$$\begin{aligned} v(x) &= K - \sqrt{x},\ 0 < x \leq L^* \\ &= Ax^{-0.5},\ L^* < x. \end{aligned}$$

The smooth pasting conditions give

$$\begin{aligned} v(L^*) &= A(L^*)^{-0.5} = K - \sqrt{L^*} \\ v_x(L^*) &= -0.5A(L^*)^{-1.5} = -\frac{1}{2\sqrt{L^*}}. \end{aligned}$$

Solving for the above system gives

$$\begin{aligned} A &= L^* \\ L^* &= \left(\frac{K}{2}\right)^2. \end{aligned}$$

Thus

$$\begin{aligned} v(x) &= K - \sqrt{x},\ 0 < x \le \left(\frac{K}{2}\right)^2 \\ &= \frac{K^2}{4} x^{-.5},\ \left(\frac{K}{2}\right)^2 < x. \end{aligned}$$

(c) (10 points) Verify that $v(x)$ that you derived in the previous part satisfy the linear complimentarity equations.
We need to verify that
(i) $v(x) = K - \sqrt{x}$ satisfies $-(0.5)v + v_x(0.5)x + v_{xx}x^2 \le 0$ on $0 < x \le \left(\frac{K}{2}\right)^2$ and
(ii) $\frac{K^2}{4}x^{-.5} \ge (K - \sqrt{x})^+$ on $\left(\frac{K}{2}\right)^2 < x$.
To verify (i), note that when $v(x) = K - \sqrt{x}$,

$$-(0.5)v + v_x(0.5)x + v_{xx}x^2 = -0.5(K - \sqrt{x}) < 0$$

because $x \le \left(\frac{K}{2}\right)^2$.
To verify (ii), note that $(K - \sqrt{x})^+ = 0$ for $x \ge K^2$ thus we only need to verify

$$v(x) = \frac{K^2}{4}x^{-.5} \ge K - \sqrt{x}$$

on $\left(\frac{K}{2}\right)^2 < x \le K^2$. On the other hand, we have

$$\frac{K^2}{4}x^{-.5} - K + \sqrt{x} = \big(\frac{K}{2x^{1/4}} - (x)^{1/4}\big)^2 \ge 0.$$

Thus $\frac{K^2}{4}x^{-.5} \ge K - \sqrt{x}$ for all x.

2. Let r, r^f, σ be constants; $r, r^f, \sigma > 0$, and $\widetilde{P}$ the domestic risk neutral measure. Let Q_t have the following dynamics under $\widetilde{P}$:

$$dQ_t = (r - r^f)\, Q_t\, dt + \sigma Q_t d\widetilde{W}(t),$$

$\widetilde{W}$ a $\widetilde{P}$ Brownian motion. Q_t is the price of 1 Euro at time t in terms of the dollars. Also assume $Q(0)$ is given.

(a) (10 points) Let V_t^E be the price of a put struck at K dollars on 1 Euro, denominated in dolars with expiry T. That is

$$V_T^E = \Big(K - Q_T\Big)^+.$$

By using Black-Scholes formula, find an explicit formula for V_0^E. (Recall the Black-Scholes formula: If

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t,$$

then

$$\widetilde{E}\Big\{e^{-rT}\big(S_T-K\big)^+\Big\} = S_0 N(d_1) - Ke^{-rT}N(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_0}{K}\right) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\log\left(\frac{S_0}{K}\right) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}) \end{aligned}$$

Ans:

$$\begin{aligned} \widetilde{E}\Big\{e^{-rT}\Big(Q(T)-K\Big)\Big\} &= e^{-r^f T}\widetilde{E}\Big\{e^{-(r-r^f)T}\Big(Q(T)-K\Big)\} \\ &= e^{-r^f T}\Big(Q(0) - e^{-(r-r^f)T}K\Big). \end{aligned}$$

Also

$$\begin{aligned} \widetilde{E}\Big\{e^{-rT}\Big(Q(T)-K\Big)^+\Big\} &= e^{-r^f T}\widetilde{E}\Big\{e^{-(r-r^f)T}\Big(Q(T)-K\Big)^+\} \\ &= e^{-r^f T}\Big(Q(0)N(\bar{d_1}) - Ke^{-(r-r^f)T}N(\bar{d_2})\Big), \end{aligned}$$

where

$$\begin{aligned} \bar{d_1} &= \frac{\log\left(\frac{Q(0)}{K}\right) + (r-r^f+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ \bar{d_2} &= \frac{\log\left(\frac{Q(0)}{K}\right) + (r-r^f-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}. \end{aligned}$$

Thus by Put-Call parity,

$$V_0^2 = e^{-r^f T}\Big(Q(0)N(\bar{d_1}) - Ke^{-(r-r^f)T}N(\bar{d_2})\Big) - e^{-r^f T}\Big(Q(0) - e^{-(r-r^f)T}K\Big).$$

(b) (10 points) Let V_t^A be the price of an American put option on Q_t with strike K and expiry T. That is the payoff function is $g(Q_t) = (K-Q_t)^+$. By Markov property, there is a function $v(t,x)$ so that $v(t,Q_t) = V_t^A$. Write down, similar to the lecture notes, the martingale characterization of $v(t,Q_t)$. Justify your answer.

Ans: Note that

$$v(t,Q_t) = V_t^A = \sup_{\tau\in[t,T]} \tilde{E}(e^{-r(\tau-t)}(K-Q_\tau)^+|\mathcal{F}_t).$$

Let $u(t,x)$ be such that

(i) $u(t,Q_t) \geq g(Q_t)\forall 0 \leq t \leq T$

(ii) $e^{-rt}u(t,Q_t)$ is a super martingale.

(iii) For any t,x let $\tau^*_{t,x} = \inf\{s \geq t : V_s = g(Q_s)|Q_t = x\} \wedge T$ then $e^{-r(t\wedge\tau^*_{t,x})}V_{t\wedge\tau^*_{t,x}}$ is a martingale.

Then $u(t,x) = V_t^A$.

To justify this, we will show $u(t, Q_t) \leq V_t^A$ and $u(t, Q_t) \geq V_t^A$.
a. $u(t, Q_t) \geq V_t^A$:
By optional sampling, for all $\tau \in [t, T]$,

$$u(t, Q_t) \geq \tilde{E}\Big(e^{-r(\tau-t)} u(\tau, Q_\tau)|\mathcal{F}_t\Big),$$

which implies

$$u(t, Q_t) \geq \tilde{E}\Big(e^{-r(\tau-t)}(K - Q_\tau)^+|\mathcal{F}_t\Big).$$

Thus

$$\begin{aligned} u(t, Q_t) &\geq \sup_{\tau \in [t,T]} \tilde{E}\Big(e^{-r(\tau-t)}(K - Q_\tau)^+|\mathcal{F}_t\Big) \\ &= V_t^A. \end{aligned}$$

b. $u(t, x) \leq v(t, x)$ (which implies $u(t, Q_t) \leq v(t, Q_t) = V_t^A$) : By the martingale property:

$$\begin{aligned} u(t, x) &= \tilde{E}(e^{-r(\tau^*_{t,x}-t)} g(u(\tau^*_{t,x}, Q_{\tau^*_{t,x}}))|S_t = x) \\ &= \leq \sup_{\tau \in [t,T]} \tilde{E}\Big(e^{-r(\tau-t)}(K - Q_\tau)^+|Q_t = x\Big) \\ &= v(t, x). \end{aligned}$$

(c) (5 points) Write down the linear complimentarity equations for $v(t, x)$ in the previous part. You only have to write down the equation and do not have to justify how you arrive at it.
Ans:

$$\begin{aligned} -rv + v_t + v_x(r - r^f)x + \frac{1}{2}v_{xx}\sigma^2 x^2 &\leq 0 \\ -rv + v_t + v_x(r - r^f)x + \frac{1}{2}v_{xx}\sigma^2 x^2 &= 0 \text{ on } v(t, x) > (K - x)^+ \\ v(t, x) &\geq (K - x)^+ \\ v(T, x) &= (K - x)^+. \end{aligned}$$

3. Consider a market with 3 risky assets whose dynamics under the physical measure are as followed:

$$\begin{aligned} dS_t^1 &= \alpha_t^1 S_t^1 dt + 2S_t^1(dW_t^1 + dW_t^3) \\ dS_t^2 &= \alpha_t^2 S_t^2 dt + S_t^2(dW_t^1 + dW_t^2) \\ dQ_t &= \alpha_t^3 Q_t dt + 3Q_t(dW_t^2 + dW_t^3). \end{aligned}$$

All assets are quoted in dollars, $W_t^i, i = 1, 2, 3$ are independent Brownian motions. We will interpret US as the domestic market and Euro as the foreign market in this question. Q_t is the price of 1 Euro in dollars. Let r and r^f be the US and Euro interet rates respectively. Also suppose that r and r^f are constants.

(a) (5 points) Does the domestic risk neutral measure exist ? If so, is it unique?
Ans: The volatility matrix is

$$\sigma = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 3 & 3 \end{bmatrix}.$$

This matrix is invertible. Therefore the risk neutral measure exists and is unique.

(b) (10 points) Suppose that $r > r^f$. An arbitrage opportunity is a self-financing portfolio π based on S^1, S^2, the foreign money market and the domestic money market such that $\pi_0 = 0, P(\pi_T \geq 0) = 1, P(\pi_T > 0) > 0$, where P is the physical probability measure. Does there exist an arbitrage opportunity? Explain.
Under the risk neutral measure $\tilde{P}$

$$\begin{aligned} dS_t^1 &= rS_t^1 dt + 2S_t^1(d\tilde{W}_t^1 + d\tilde{W}_t^3) \\ dS_t^2 &= rS_t^2 dt + S_t^2(d\tilde{W}_t^1 + d\tilde{W}_t^2) \\ dM_t^f &= rM_t^f dt + 3M_t^f(d\tilde{W}_t^2 + d\tilde{W}_t^3). \end{aligned}$$

If V is a self-financing portfolio that invests in Δ_t^i shares for S^i, Δ_t shares in the foreign money market M^f and the rest in the domestic money market we have

$$dV_t = \Delta_t^1 dS_t^1 + \Delta_t^2 dS_t^2 + \Delta_t dM_t^f + (V_t - [\Delta_t^1 S_t^1 + \Delta_t^2 S_t^2 + \Delta_t M_t^f]r)dt.$$

Plug in the dynamics of dS_t^i and dM_t^f as given above gives

$$dV_t = rV_t dt + (\cdots d\tilde{W}_t^1 + \cdots d\tilde{W}_t^2 + \cdots d\tilde{W}_t^3).$$

That is

$$d(D_t V_t) = \cdots d\tilde{W}_t^1 + \cdots d\tilde{W}_t^2 + \cdots d\tilde{W}_t^3.$$

It follows that $D_t V_t$ is a $\tilde{P}$ martingale. Since the risk neutral measure $\tilde{P}$ is equivalent to the physical measure P, the condition that $V_0 = 0, P(V_T \geq 0) = 1, P(V_T > 0) > 0$ cannot be satisfied. Thus an arbitrage opportunity does not exist.

(c) (10 points) Consider a contract that allows the contract holder to exchange 1 share of S^1 and 1 share of S^2 at time T for a fixed $K(t, T)$ **Euro**, $K(t, T)$ is to be determined at the time $t < T$ when the contract is written. We also require that there is no cost to enter the contract. Determine $K(t, T)$.

The value of the contract at time T (in dollars) is $V_T = S_T^1 + S_T^2 - K(t,T)Q(T)$. We will use the foreign money market numéraire to price this contract.
Denote $S_t^{Nf,1} := \frac{S_t^1}{N_t^f}, S_t^{Nf,2} := \frac{S_t^2}{N_t^f}$ we have

$$V_T^{Nf} = S_T^{Nf,1} + S_T^{Nf,2} - D_T^f K(t,T).$$

Since $V_t = 0, V_t^{Nf} = 0$. Thus

$$0 = \tilde{E}^{Nf}\Big(S_T^{Nf,1} + S_T^{Nf,2} - D_T^f K(t,T)|\mathcal{F}_t\Big).$$

Since under $P^{Nf}, S^{Nf,1}, S^{Nf,2}$ are martingales, r^f is a constant, the above implies that

$$K(t,T) = e^{r^f T}(S_t^{Nf,1} + S_t^{Nf,2}).$$

Written in terms of S^1, S^2, Q_t we have

$$K(t,T) = e^{r^f(T-t)}(\frac{S_t^1}{Q_t} + \frac{S_t^2}{Q_t}).$$

4. At time t, let Q_t^{EU} be the price of 1 Euro in US dollars and Q_t^{CU} be the price of 1 Chinese Yuan in US dollars. Let r_t^U, r_t^E, r_t^C be the interest rates of the US, Euro and China money market respectively. Suppose their dynamics under the physical measure are as followed:

$$\begin{aligned} dQ_t^{EU} &= \alpha_t^1 Q_t^{EU} dt + Q_t^{EU}\sigma^1 dW_t^1 \\ dQ_t^{CU} &= \alpha_t^2 Q_t^{CU} dt + Q_t^{CU}\sigma^2(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2). \end{aligned}$$

We assume here that $\sigma^i, i = 1,2$ and ρ are constants, $0 < \rho < 1$, $W^i, i = 1,2$ are independent Brownian motions.

(a) (5 points) Let P^U be the risk neutral measure associated with the choice of the US money market as numéraire. Assuming P^U exists, write down the dynamics of Q_t^{EU} and Q_t^{CU} under P^U.
Ans:

$$\begin{aligned} dQ_t^{EU} &= (r_t^U - r_t^E)Q_t^{EU} dt + Q_t^{EU}\sigma^1 dW_t^{U,1} \\ dQ_t^{CU} &= (r_t^U - r_t^C)Q_t^{CU} dt + Q_t^{CU}\sigma^2(\rho dW_t^{U,1} + \sqrt{1-\rho^2}dW_t^{U,2}), \end{aligned}$$

where $W^{U,i}, i = 1,2$ are Brownian motions under P^U.

(b) (5 points) Does P^U exist? If so, is it unique?
If $0 < \rho < 1$ the matrix

$$\begin{bmatrix} \sigma^1 & 0 \\ \rho\sigma^2 & \sqrt{1-\rho^2}\sigma^2 \end{bmatrix}$$

is invertible, P^U exists and is unique.

(c) (5 points) Let Q^{EC} be the price of 1 Euro in Chinese Yuan and P^C be the risk neutral measure associated with the Chinese money market. Find the dynamics of Q^{EC} under P^C.

Ans: $Q_t^{EC} = \frac{Q_t^{EU}}{Q_t^{CU}}$. Thus

$$dQ_t^{EC} = (r_t^C - r_t^E)Q_t^{EC}dt + Q_t^{EC}\Big((\sigma^1 - \rho\sigma^2)dW_t^{C,1} - \sigma^2\sqrt{1-\rho^2}dW_t^{C,2}\Big),$$

where $W^{C,i}, i = 1, 2$ are Brownian motions under P^C.

(d) (10 points) Let $V_T^1 = (Q_T^{CE} - K)^+$ (Euro) be the value of the call option that allows the option holder to buy 1 yuan with K Euros at time T. Let $V_T^2 = (\frac{1}{K} - Q_T^{EC})^+$ (yuan) be the value of a put option that allows the option holder to sell 1 Euro for $\frac{1}{K}$ yuan at time T. Find the relation between V_0^1 and V_0^2.

Ans:

We will denote the discount process for the US money market as

$$D_t^U := e^{-\int_0^t r_s^U ds},$$

and the value of 1 unit of the US money market at time t is

$$M_t^U := e^{\int_0^t r_s^U ds}.$$

We denote similarly for $D_t^E, M_t^E, D_t^C, M_t^C$. Then

$$\begin{aligned}
V_0^1 &= \tilde{E}^E\big(D_T^E(Q_T^{CE} - K)^+\big) \\
&= \tilde{E}^C\big(\frac{Q_0^{CE}}{Q_T^{CE}M_T^C D_T^E} D_T^E(Q_T^{CE} - K)^+\big) \\
&= Q_0^{CE}\tilde{E}^C\big(D_T^C(1 - \frac{K}{Q_T^{CE}})^+\big) \\
&= KQ_0^{CE}\tilde{E}^C\big(D_T^C(\frac{1}{K} - Q_T^{EC})^+\big) \\
&= KQ_0^{CE}V_0^2.
\end{aligned}$$