Math 622
Spring 2015
Midterm 1 - Form A
03/04/2015

Name (Print): ____________________

This exam contains 13 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- The parts of the problems are **not necessarily connected.** If you cannot do one part, you can assume the result of that part, if needed, to do the next part.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	25	
3	25	
4	20	
5	10	
Total:	100	

1. In the wake of the recent cyberattack, Sony wants to understand the impact of such attacks on its stock price. The financial experts at Sony come up with the following model for Sony stock price, under the risk neutral measure $\mathbb{Q}$:

$$dS_t = rS_t dt + \sigma S_t dW_t - \gamma S_{t-} d(N_t - \lambda t),$$

where N_t is a Poisson process with rate λ, λ is the frequency of the attack and $0 < \gamma < 1$ is the percentage loss in the stock price after the attack. $r, \sigma, \gamma, \lambda$ are constants.

(a) (5 points) Let τ be an Exponential (λ) random variable. That is τ has p.d.f.

$$f_\tau(t) = \lambda e^{-\lambda t}, t \geq 0.$$

Let W_t be a Brownian motion independent of τ. By conditioning on τ (or using other techniques of your choice), compute $E(e^{\sigma W_\tau})$. What value must σ take so that the expectation is finite?

Ans:

$$\begin{aligned} E(e^{\sigma W_\tau}) &= \int_0^\infty E(e^{\sigma W_\tau} | \tau = t) f_\tau(t) dt \\ &= \int_0^\infty e^{\frac{1}{2}\sigma^2 t} \lambda e^{-\lambda t} dt \\ &= \frac{2\lambda}{2\lambda - \sigma^2}. \end{aligned}$$

We need $\frac{1}{2}\sigma^2 < \lambda$ for the integral to be finite.

(b) (5 points) Let 0 represent the current time. Let τ be the first time after 0 that Sony experiences a cyberattack. That is

$$\tau := \inf\{t \geq 0 : N_t = 1\}.$$

Compute the expected change in Sony stock price after the attack. That is compute $E^{\mathbb{Q}}(e^{-r\tau}\Delta S_\tau)$.

Ans:

$$\Delta S_\tau = -\gamma S_{\tau-} = -\gamma S_0 e^{(r+\gamma\lambda-\frac{1}{2}\sigma^2)\tau+\sigma W_\tau}.$$

Therefore

$$\begin{aligned} E^{\mathbb{Q}}(e^{-r\tau}\Delta S_\tau) &= E^{\mathbb{Q}}(-\gamma S_0 e^{(\gamma\lambda-\frac{1}{2}\sigma^2)\tau+\sigma W_\tau}) \\ &= \int_0^\infty -\gamma S_0 e^{(\gamma\lambda-\frac{1}{2}\sigma^2)t+\frac{1}{2}\sigma^2 t}\lambda e^{-\lambda t}dt \\ &= -\frac{S_0\gamma}{1-\gamma}, \end{aligned}$$

by conditioning on τ as above.

(c) (5 points) Let V_t represent the price of a forward contract on Sony's stock with strike K expiry T. that is $V_T = S_T - K$. Compute V_t.
Ans: $V_t = E^{\mathbb{Q}}(e^{-r(T-t)}(S_T - K)|\mathcal{F}_t) = S_t - e^{-r(T-t)}K$.

(d) (5 points) Let τ be the first time after 0 that Sony experiences a cyberattack as above. We want to understand the effect of a cyberattack on the price of the forward contract. Since the contract has expiry T, only the stopping time $\tau \wedge T$ is relevant. Compute

$$E^{\mathbb{Q}}\Big(e^{-r(\tau\wedge T)}\Delta V_{\tau\wedge T}\Big).$$

Ans:

$$\Delta V_{\tau\wedge T} = \Delta S_{\tau\wedge T} = -\gamma e^{(r+\gamma\lambda-\frac{1}{2}\sigma^2)(\tau\wedge T)+\sigma W_{\tau\wedge T}}.$$

Therefore

$$\begin{aligned} E^{\mathbb{Q}}\Big(e^{-r(\tau\wedge T)}\Delta V_{\tau\wedge T}\Big) &= \int_0^T -\gamma S_0 e^{(\gamma\lambda-\frac{1}{2}\sigma^2)t+\frac{1}{2}\sigma^2 t}\lambda e^{-\lambda t}dt \\ &= -\frac{S_0\gamma}{1-\gamma}(1-e^{\lambda(\gamma-1)T}). \end{aligned}$$

2. A Math 622 student, Mr. Brown, just got an offer from Goldman Sach to model an asset with jumps. In the over-excitement of getting the job, Mr. Brown forgot to use S_{t-} on the right hand side of the equation for dS_t in his model. That is he comes up with the following model, under the physical measure $\mathbb{P}$:

$$\begin{aligned} dS_t &= \sigma S_t dW_t + \gamma S_t d(N_t - \lambda t), \\ S_0 &= 1. \end{aligned}$$

where N is a Poisson process with rate λ. γ, λ are constants, $\gamma \neq 0$.

(a) (5 points) Solve explicitly for S_t and justify your solution by Ito's formula. What must γ be so that $S_t > 0$ for all t?

Ans:

$$S_t = e^{(-\frac{1}{2}\sigma^2 - \gamma\lambda)t + \sigma W_t} \prod_{i=1}^{N_t} \frac{1}{1-\gamma}.$$

We must have $\gamma < 1$ so that $S_t > 0$.

Check by Ito's formula:

$$S_t = e^{(-\frac{1}{2}\sigma^2 - \gamma\lambda)t + \sigma W_t + \sum_{i=1}^{N_t} \log\left(\frac{1}{1-\gamma}\right)}.$$

At each jump point u of S_t, the jump size is

$$\begin{aligned} \Delta S_u &= e^{(-\frac{1}{2}\sigma^2 - \gamma\lambda)u + \sigma W_u + \sum_{i=1}^{N_u} \log\left(\frac{1}{1-\gamma}\right)} - e^{(-\frac{1}{2}\sigma^2 - \gamma\lambda)u + \sigma W_u + \sum_{i=1}^{N_u - 1} \log\left(\frac{1}{1-\gamma}\right)} \\ &= e^{(-\frac{1}{2}\sigma^2 - \gamma\lambda)u + \sigma W_u + \sum_{i=1}^{N_u} \log\left(\frac{1}{1-\gamma}\right)} (1 - e^{-\log\left(\frac{1}{1-\gamma}\right)}) \\ &= S_u(1 - (1-\gamma)) = S_u\gamma. \end{aligned}$$

Thus

$$\begin{aligned} S_t &= S_0 + \int_0^t -\gamma\lambda S_u du + \int_0^t \sigma S_u dW_u + \sum_{0<u\leq t} S_u\gamma \\ &= S_0 + \int_0^t \sigma S_u dW_u + \int_0^t S_u \gamma d(N_t - \lambda t). \end{aligned}$$

(b) (5 points) Prove that S_t is not a $\mathbb{P}$ martingale.

$$\begin{aligned} dS_t &= \sigma S_t dW_t + \gamma S_t d(N_t - \lambda t) \\ &= \sigma S_t dW_t + \gamma \frac{1}{1-\gamma} S_{t-} dN_t - \lambda\gamma S_t dt \\ &= \sigma S_t dW_t + \gamma \frac{1}{1-\gamma} S_{t-} d(N_t - \lambda t) + \frac{\lambda\gamma^2}{1-\gamma} S_t dt \end{aligned}$$

Since $S_t > 0$ and $\frac{\lambda\gamma^2}{1-\gamma} > 0$ clearly the dt term of dS_t is not 0. Thus S_t is not a $\mathbb{P}$-martingale. Remark: $S_t = \frac{1}{1-\gamma} S_{t-}$ only at the jump points of N_t. That's why we need to separate the dN_t and the dt term in the second equality above.

(c) (5 points) A risk neutral measure $\mathbb{Q}$ is such that $e^{-rt} S_t$ is a $\mathbb{Q}$-martingale. Does a risk neutral measure exist in Mr. Brown model? If so, under what condition?

We can clearly choose a measure $\mathbb{Q}$ such that

$$\widetilde{W_t} := W_t + \frac{\lambda\gamma^2}{\sigma(1-\gamma)} t$$

is a $\mathbb{Q}$-Brownian motion and N_t is a $\mathbb{Q}$-Poisson process with rate $\lambda + r$. Then the dynamics of S_t under the new measure $\mathbb{Q}$ will be

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t + \gamma S_{t-} d(N_t - (\lambda + \frac{r}{\gamma})t).$$

Thus $e^{-rt} S_t$ is a $\mathbb{Q}$-martingale. In other words, a risk neutral measure always exists in Mr. Brown's model.

(d) (10 points) Let N^1, N^2 be two independent Poisson processes with rate λ_1, λ_2 respectively. Is the process

$$X_t := \int_0^t N_t^2 d(N_t^1 - \lambda_1 t)$$

a martingale? (Note: it is N_t^2 in the integrand, not N_{t-}^2). Justify your answer.

Ans:

$$\begin{aligned} X_t &= \int_0^t N_u^2 d(N_u^1 - \lambda_1 u) \\ &= \int_0^t (N_{u-}^2 + \Delta N_u^2) d(N_u^1 - \lambda_1 u) \\ &= \int_0^t N_{u-}^2 d(N_u^1 - \lambda_1 u) + \int_0^t \Delta N_u^2 d(N_u^1 - \lambda_1 u). \end{aligned}$$

Since N^1, N^2 are independent, they cannot jump at the same time. That is

$$\int_0^t \Delta N_u^2 dN_u^1 = 0.$$

Moreover, since N_t^2 only jumps finitely many times on any finite interval

$$\int_0^t \Delta N_u^2 du = 0,$$

as well. That is

$$X_t = \int_0^t N_{u-}^2 d(N_u^1 - \lambda_1 u).$$

Thus X_t is a martingale.

3. (a) (5 points) Let W^1, W^2 be two Brownian motions such that their quadratic covariation $\langle W^1, W^2\rangle_t$ is 0 for all t. That is for any $f \in C^{1,2,2}$

$$\begin{aligned} df(t, W_t^1, W_t^2) &= [(f_t dt + \frac{1}{2}f_{xx} + \frac{1}{2}f_{yy})(t, W_t^1, W_t^2)]dt \\ &+ f_x(t, W_t^1, W_t^2)dW_t^1 + f_y(t, W_t^1, W_t^2)dW_t^2. \end{aligned}$$

Are W^1, W^2 independent? Justify your answer. (A quote from some sources will not be accepted. You'll need to come up with your own justification here).

Ans: W^1, W^2 are independent. It is sufficient to show

$$X_t := e^{u_1 W_t^1 + u_2 W_t^2 - \frac{1}{2}u_1^2 t - \frac{1}{2}u_2^2 t}$$

is a martingale for all u_1, u_2. Since then we have

$$e^{u_1 W_t^1 + u_2 W_t^2} = e^{\frac{1}{2}u_1^2 t} e^{\frac{1}{2}u_2^2 t},$$

and W^1, W^2 are independent following Kac's lemma.

Now X_t is a martingale because

$$dX_t = u_1 X_t dW_t^1 + u_2 X_t dW_t^2.$$

(b) (10 points) Consider a market with three assets S^1, S^2, S^3 with the following dynamics under *the physical measure* $\mathbb{P}$:

$$\begin{aligned} dS_t^1 &= S_t^1 dt + S_t^1 dW_t^1 \\ dS_t^2 &= S_t^2 dt + S_t^2 dW_t^1 + S_t^2 dW_t^2 \\ dS_t^3 &= S_t^3 dt + S_t^3 dW_t^2 + S_{t-}^3 d(Q_t - \mu t), \end{aligned}$$

where W^1, W^2, Q are independent,

$$Q_t = \sum_{i=0}^{N_t} Y_i,$$

Y_i are i.i.d discrete random variables with distribution

$$P(Y_i = 1/2) = 2/3; P(Y_i = 1) = 1/3,$$

N_t is a Poisson(1) process, Y_i and N are independent. $\mu = E^{\mathbb{P}}(Q_1)$.
A risk neutral measure $\mathbb{Q}$ is such that $e^{-rt} S_t^i$ is a martingale, $i = 1, 2, 3$. *Suppose that*

$$0 < r < 0.01.$$

Does a risk neutral measure exist for this market? If so, under what condition(s)? If the risk neutral measure exists, is it unique? If so, under what condition(s)?
Ans: Observe that $E(Y_1) = 2/3$. Thus $\mu = E^{\mathbb{P}}(Q_1) = 2/3$.
We see that if a risk neutral measure $\mathbb{Q}$ exists, necessarily $W_t^1 + (1-r)t$ is a $\mathbb{Q}$-Brownian motion (from requiring that $e^{-rt} S_t^1$ is a $\mathbb{Q}$ martingale). It then follows that W_t^2 is necessarily a $\mathbb{Q}$-Brownian motion (from requiring that $e^{-rt} S_t^2$ is a $\mathbb{Q}$ martingale).
We can write dS_t^3 as

$$dS_t^3 = r S_t^3 dt + S_t^3 dW_t^2 + S_{t-}^3 d(Q_t - (\mu - 1 + r)t).$$

Thus $e^{-rt} S_t^3$ is a $\mathbb{Q}$-martingale only if

$$E^{\mathbb{Q}}(Q_1) = \mu - 1 + r < 2/3 - 1 + 0.01 < 0.$$

On the other hand $E^{\mathbb{Q}}(Y_1)$ must be positive since Y_1 takes only positive values. The new rate $\tilde{\lambda}$ of N_t under $\mathbb{Q}$ must also be positive. Thus $E^{\mathbb{Q}}(Q_1) = \tilde{\lambda} E^{\mathbb{Q}}(Y_1) > 0$. It follows that a risk neutral measure does not exist for this market.

(c) (10 points) (This question is independent of the above) Let S be an asset with the following dynamics under *the risk neutral measure* $\mathbb{Q}$:

$$dS_t = rS_t dt + S_t dW_t + S_{t-} d(Q_t - \mu t),$$

where W, Q are independent,

$$Q_t = \sum_{i=0}^{N_t} Y_i,$$

Y_i are i.i.d discrete random variables with distribution

$$P(Y_i = 1/2) = 2/3; P(Y_i = 1) = 1/3,$$

N_t is a Poisson(λ) process, Y_i and N are independent. $\mu = E^{\mathbb{Q}}(Q_1)$.
Write down the PIDE for the value V_t at time t of a call option on S_t with strike K and expiry T.
Ans: $V_t = c(t, S_t)$ where $c(t,x)$ satisfies:

$$\begin{aligned}
-rc(t,x) &+ \frac{\partial}{\partial t}c(t,x) + (r-\mu)x\frac{\partial}{\partial x}c(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}c(t,x)\sigma^2 x^2 \\
&+ 2/3\lambda[c(t,x(1+1/2)) - c(t,x)] \\
&+ 1/3\lambda[c(t,x(1+1)) - c(t,x)] = 0, 0 \le t < T, x > 0; \\
c(T,x) &= (x-K)^+, x > 0.
\end{aligned}$$

4. (20 points) Inspired by the chapter on exotic options, Mr. Brown wants to introduce even more exotic option to his company. In particular, he comes up with a so-called lookback Asian option, as followed. Let S_t have Black-Scholes dynamics under *a risk neutral measure* $\mathbb{Q}$:

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t \\ S_0 &= x. \end{aligned}$$

We say V is a lookback Asian option with expiry T if it pays at time T

$$V_T = \max_{u\in[0,T]} \int_0^u S_s ds.$$

(a) Let $Y_t = \int_0^t S_u du$. Let V_t be the value of the lookback Asian option at time t. Investigate whether we can find a function $v(t,x,y)$ such that $v(t,S_t,Y_t) = V_t$ (possibly up to some stopping time). If so, find the PDE that $v(t,x,y)$ has to satisfy.

Ans: Observe that $V_T = \int_0^T S_t dt$ so this is just a regular Asian option. We have

$$\begin{aligned} V_t &= E^{\mathbb{Q}}(e^{-r(T-t)} \int_0^T S_t dt|\mathcal{F}_t) \\ &= e^{-r(T-t)} E^{\mathbb{Q}}(Y_t + \int_t^T S_u du|\mathcal{F}_t) \\ &= v(t,x,y), \end{aligned}$$

where

$$v(t,x,y) = e^{-r(T-t)} E\Big(y + \int_t^T x e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W_u - W_t)} du\Big).$$

It is actually easier to answer part b first: find $v(t,x,y)$. We have

$$\begin{aligned} v(t,x,y) &= e^{-r(T-t)} E\Big(y + \int_t^T x e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W_u-W_t)} du\Big) \\ &= e^{-r(T-t)} \Big(y + \int_t^T x e^{(r-\frac{1}{2}\sigma^2)(u-t)} E(e^{\sigma(W_u-W_t)}) du\Big) \\ &= e^{-r(T-t)} \Big(y + \int_t^T x e^{r(u-t)} du\Big) \\ &= e^{-r(T-t)} \Big(y + x\frac{e^{r(T-t)}-1}{r}\Big). \end{aligned}$$

Thus the PDE is

$$\begin{aligned} & -rv + v_t + v_x rx + v_y x + \frac{1}{2} v_{xx}\sigma^2 x^2 = 0, \\ & 0 < x, y < \infty, 0 \le t < T; \\ & v(T,x,y) = y; \\ & v(t,0,y) = e^{-r(T-t)} y; \\ & v(t,x,0) = e^{-r(T-t)} x \frac{e^{r(T-t)}-1}{r}. \end{aligned}$$

(b) Find $v(t, x, y)$

Ans: See part a.

5. (10 points) Before finishing his tenure with Golman Sach, Mr. Brown introduced to his company one last exotic option, called a perpetual two-barrier option, as followed. Let S_t have Black-Scholes dynamics under *the physical measure* $\mathbb{P}$:

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t \\ S_0 &= x. \end{aligned}$$

Let $0 < A < x < B$ be two given constants. The perpetual two-barrier option would pay the option holder B dollars if S_t hits B before A and pay nothing if S_t hits A before B. This option has no expiration date. You can assume that the probability S_t will hit A or B eventually is 1 (that is the probability that S_t fluctuates forever between A and B without hitting either barrier is 0).

Suppose the interest rate $r = 0$. Use the following version of the optional sampling theorem:

If X_t is a martingale and τ is a stopping time such that $|X_{t\wedge\tau}| \leq K$ for some constant K then $E(X_\tau) = X_0$

to find the risk-netral price at time 0, V_0, for the above perpetual two-barrier option.

Ans: Let

$$\tau := \inf\{t : S_t = A \text{ or } S_t = B\}.$$

Then τ is a stopping time and $|S_{t\wedge\tau}| \leq \max(A, B)$. Under the risk neutral measure, since $r = 0$, S_t is a martingale. Thus by the optional sampling theorem

$$E^{\mathbb{Q}}(S_\tau) = S_0 = x.$$

But $S_\tau = B$ with probability p and $S_\tau = A$ with probability $1 - p$. Therefore

$$(1 - p)A + pB = x,$$

or

$$p = \frac{x - A}{B - A}.$$

The price of the option is:

$$V_0 = E^{\mathbb{Q}}(B\mathbf{1}_{S_\tau = B}) = B\frac{x - A}{B - A}.$$