Change of numéraire

Math 622

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1 Introduction

1.1 Another look at the Black-Scholes risk neutral model

Let $r > 0$ be the constant risk free rate. So far, we've considered the following Black-Scholes model of a stock:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

where α is a constant and W_t a Brownian motion.
To price any financial derivative based on S, the first question we have to answer is: what is the risk neutral probability measure? In other words, we want to find a probability measure $\widetilde{P}$ such that $e^{-rt}S_t$ is a martingale under $\widetilde{P}$.
We're used to looking at $e^{-rt}S_t$ as the discounted stock price. And the risk neutral measure is interpreted as the probability such that the discounted stock price is a martingale.
There is yet a slightly different way of looking at this. If we denote

$$\begin{aligned} dNt &= rNtdt \\ N_0 &= 1, \end{aligned}$$

that is $N_t = e^{rt}$; then N_t is the price of one unit of the *money market account.* Then $e^{-rt}S_t$ is nothing but the price of the stock *expressed in the unit of the money market account.* The risk neutral measure above can be looked at as the probability such that the price of the stock, *expressed in the unit of the money market account*, is a martingale.
Note that there is another asset, which is also a martingale (albeit a trivial one), when expressed in the unit of the money market account: the money market price

process itself. It is clear that the price of the money market is 1 when expressed under its own unit, thus it is a (trivial) martingale.

1.2 Main questions of this chapter

The process N_t in the above is a *numéraire*, and the risk neutral measure we've studied in the Black-Scholes model is the risk neutral measure associated to the (domestic) money market numéraire. To re-emphasize, it is the probability measure such that the price of all non-dividend paying assets are martingales when expressed in the unit of the domestic money market account.

It is clear that the domestic money market is not the only choice for a numéraire. In a world where there is a foreign currency, then the foreign money market is also a possible choice of numéraire. The obvious question is, *how do we determine the risk neutral probability associated with the foreign money market numéraire*? More generally, how do we decide a risk neutral probability associated with any numéraire, as long as we have a model for that particular numéraire?

Specifically, letting

$$D_t := e^{-\int_0^t R_u du},$$

be the discount process, and suppose we have two underlying assets S_t, N_t so that both $D_t N_t$ and $D_t S_t$ are martingales under the risk neutral measure $\tilde{P}$. Denoting

$$S_t^{(N)} := \frac{S_t}{N_t}$$

as the "price" of S_t under the numéraire N_t, we will address the following questions:

a. Does there exist a measure $\widetilde{P}^{(N)}$ so that $S_t^{(N)}$ is a martingale under $\widetilde{P}^{(N)}$? If yes, we will call $\widetilde{P}^{(N)}$ the risk neutral measure associated with the numéraire N_t.

b. What is the dynamics of $S_t^{(N)}$ under $\widetilde{P}^{(N)}$?

c. How can we relate the pricing of a derivative V_t based on S_t and N_t under the risk neutral measure $\widetilde{P}$ with the pricing of

$$V_t^{(N)} := \frac{V_t}{N_t}$$

under the measure $\widetilde{P}^{(N)}$?

One important thing to note about the choice of numéraire: we shall take only *non-dividend* paying assets as numéraire. Another way to put it is that we will only use asset N_t that satisfies $D_t N_t$ is a martingale under the risk neutral measure as a numéraire. Using this criterion, the domestic currency itself **cannot** be used as a numéraire because its value stays constant for all time t.

1.3 New set up of this chapter

Since in this chapter, we will introduce the foreign exchange rate and foreign money market, it is natural that we are into a multiple risky assets setting. Moreover, the risk free rate *will no longer be a constant* r. We will consider a risk-free rate process $R(t)$ that can be stochastic. The associated discount process is

$$D_t = \exp\Big\{ -\int_0^t R(u)du \Big\},$$

and for the domestic money market risk neutral measure $\widetilde{P}$, we will require that $D_t S_t$ be a martingale under $\widetilde{P}$. Also, as we use different numéraires, there will be different risk-neutral measures corresponding to these numéraires. It is important to clarify which risk-neutral probability we are discussing. For example, we will call the risk neutral measure associated with the domestic money market *the domestic risk neutral measure.* Similarly, we will call the risk neutral measure associated with the foreign money market *the foreign risk neutral measure.* In this note, by dollars we also mean the domestic currency and vice versa.
To prepare for these new set ups, we will review a few details on stochastic calculus and multi-asset model in the next couple sections.

1.4 Why study change of numéraire

(i) A risk neutral pricing formula when the financial product is quoted in foreign currency:
Suppose we have a Euro style derivative that pays V_T (in foreign currency) at time T. We want to find the no-arbitrage price V_0 of this derivative at time 0. Let $R^f(t)$ be the foreign interest rate, which is an adapted process. Intuitively, the pricing formula would be

$$V_0 = \widetilde{E}^f \Big[e^{-\int_0^T R^f(u)du} V_T \Big],$$

where $\widetilde{E}^f$ is a foreign risk neutral measure. How to define this $\widetilde{E}^f$ so that the above formula holds is a question that we will address in this chapter.

(ii) Modeling when the interest rate is random:

When the interest rate is random, the pricing formula for a Euro-style derivative on a stock S_t becomes complicated (See formula 9.4.6 in Shreve and the discussion after). However, when we use a suitable numéraire, which is the zero-coupon bond in this case, the pricing formula becomes much simpler (formula 9.4.7 in Shreve). Thus this suggests one should model S_t under the risk-neutral measure associated with the zero-coupond bond (called the T-forward measure). Indeed, it turns out that the correct object to model is the forward price of the stock S_t (Section 9.4.3 in Shreve). The point is that our usual choice of numéraire (the domestic money market) may not be the best choice in all situations. Studying the change of numéraire suggests other choice of numéraire that would simplify the problem, both in terms of pricing and in terms of modeling.

2 Markets with multiple risky assets

Itô process models for markets with multiple risky assets are treated in Chapter 5 of Shreve. This is a brief review.
Consider a market with m risky assets. Prices are given in a domestic currency, which, for convenience, we will assume to be US dollars. A price model consists of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a filtration $\{\mathcal{F}(t);\, t \geq 0\}$, and an m-vector-valued stochastic process $S(t) = (S_1(t), \ldots, S_m(t))$ that represents the asset prices and that is adapted to the filtration. The goal of modeling is to construct S so that its statistical behavior approximately matches what is actually observed in the market. The return of asset i over the small interval of time $[t, t+dt]$ is given by $\dfrac{dS_i(t)}{S_i(t)}$.
From analysis of the historical data or from structural models for the market, the modeler can generate estimates for all assets of
(i) the mean rate, local of change of asset i: $\mu_i(t)\,dt = E\left[\dfrac{dS_i(t)}{S_i(t)} \,\Big|\, \mathcal{F}(t)\right]$;
(ii) the local (square) volatility of asset i: $\sigma_i^2(t)\,dt = \mathrm{Var}\left(\dfrac{dS_i(t)}{S_i(t)} \,\Big|\, \mathcal{F}(t)\right)$; and
(iii) the correlation between the returns of asset i and j, $i \neq j$;

$$\rho_{ij}(t), dt = \frac{\mathrm{Cov}\left(\frac{dS_i(t)}{S_i(t)}, \frac{dS_j(t)}{S_j(t)} \,\Big|\, \mathcal{F}(t)\right)}{\sigma_i(t)\sigma_j(t)}.$$

A nice way to construct models that can fit these informally described parameters, and for which the price processes are continuous, is to use stochastic differential equations driven by a multi-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_d(t))$. Suppose we set

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)\,dt + \sum_{k=1}^{d} \sigma_{ik}(t)\,dW_k(t), \qquad 1 \le i \le m. \tag{1}$$

Let $\{\mathcal{F}(t);\, t \ge 0\}$ be a filtration for W and assume $\mu_i(t)$ and $\sigma_{ij}(t)$, $t \ge 0$ are adapted to $\{\mathcal{F}(t);\, t \ge 0\}$. Recall that, by definition, $W_1, \ldots, W_d$ are independent Brownian motions. Then, formally, $E[dW_i(t) \mid \mathcal{F}(t)] = 0$, $E[(dW_i(t))^2 \,\Big|\, \mathcal{F}(t)] = dt$ and $E[dW_i(t))\,dW_j(t) \,\Big|\, \mathcal{F}(t)] = 0$. Thus, for each asset,

$$E\left[\frac{dS_i(t)}{S_i(t)} \,\Big|\, \mathcal{F}(t)\right] = \mu_i(t)\,dt + \sum_{k=1}^{d} \sigma_{ik}(t) E[dW_k(t) \,\Big|\, \mathcal{F}(t)] = \mu_i(t)\,dt,$$

in conformity with (i). On the other hand,

$$\begin{aligned} \operatorname{Var}\left(\frac{dS_i(t)}{S_i(t)} \,\Big|\, \mathcal{F}(t)\right) &= E\left[\left(\sum_{k=1}^{d} \sigma_{ik}(t)\,dW_k(t)\right)^2 \,\Big|\, \mathcal{F}(t)\right] \\ &= \left[\sum_{k=1}^{d} \sigma_{ik}^2(t)\right] dt. \end{aligned} \tag{2}$$

By a similar calculation

$$\begin{aligned} \operatorname{Cov}\left(\frac{dS_i(t)}{S_i(t)}, \frac{dS_j(t)}{S_j(t)} \,\Big|\, \mathcal{F}(t)\right) &= E\left[\sum_{k=1}^{d} \sigma_{ik}(t)\,dW_k(t) \cdot \sum_{l=1}^{d} \sigma_{jl}(t)\,dW_l(t) \,\Big|\, \mathcal{F}(t)\right] \\ &= \left[\sum_{k=1}^{d} \sigma_{ik}(t)\sigma_{jk}(t)\right] dt. \end{aligned} \tag{3}$$

Therefore, we can match the model (1) to the variances and correlations prescribed in (ii) and (iii) by choosing d and $\sigma_{ij}(t)$, $1 \le i, j \le m$, so that

$$\begin{aligned} \sum_{k=1}^{d} \sigma_{ik}^2(t) &= \sigma_i^2(t) \\ \sum_{k=1}^{d} \sigma_{ik}(t)\sigma_{jk}(t) &= \rho_{ij}(t)\sigma_i(t)\sigma_j(t) \end{aligned}$$

Because model (1) is flexible enough to capture price means, volatilities, and correlations in this manner, it is a standard model for multi-asset markets. Usually, it is expressed in the more familiar form

$$dS_i(t) = \mu_i(t)S_i(t)\,dt + S_i(t)\sum_{k=1}^{d}\sigma_{ik}(t)\,dW_k(t). \tag{4}$$

Example 1. Given $\sigma_1(t)$, $\sigma_2(t)$, and $\rho(t)$ satisfying $-1 \le \rho(t) \le 1$, we want to construct a model with two risky assets so that the volatility process of S_1 is $\sigma_1(t)$, that of S_2 is $\sigma_2(t)$ and

$$\rho(t) = \frac{\mathrm{Cov}\left(\frac{dS_1(t)}{S_1(t)}, \frac{dS_2(t)}{S_2(t)}\right)}{\sigma_1(t)\sigma_2(t)}.$$

This can be achieved with the model

$$\begin{aligned} dS_1(t) &= \mu_1(t)S_1(t)\,dt + \sigma_1(t)S_1(t)\,dW_1(t), & (5)\\ dS_2(t) &= \mu_2(t)S_2(t)\,dt + \sigma_2(t)S_2(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]. & (6)\end{aligned}$$

Indeed, equation (5) is by itself just the usual model for an asset with volatility process $\sigma_1(t)$. As for S_2,

$$\begin{aligned} &\mathrm{Var}\left(\frac{dS_2(t)}{S_2(t)} \;\Big|\; \mathcal{F}(t)\right) = E\left[\left(\sigma_2(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]\right)^2 \;\Big|\; \mathcal{F}(t)\right] \\ &= \sigma_2^2(t)\left[\rho^2(t)E[(dW_1(t))^2] + (1-\rho^2(t))E[(dW_2(t))^2] + 2\rho(t)\sqrt{1-\rho^2(t)}E[dW_1(t)\,dW_2(t)]\right] \\ &= \sigma_2^2(t)\left[\rho^2(t)\,dt + (1-\rho^2(t))\,dt\right] = \sigma_2^2(t)\,dt \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{Cov}\left(\frac{dS_1(t)}{S_1(t)}, \frac{dS_2(t)}{S_2(t)} \;\Big|\; \mathcal{F}(t)\right) &= E\left[\sigma_1(t)\,dW_1(t)\cdot\left(\sigma_2(t)\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right) \;\Big|\; \mathcal{F}(t)\right] \\ &= \sigma_1(t)\sigma_2(t)\left[\rho(t)E[(dW_1(t))^2] + \sqrt{1-\rho^2(t)}E[dW_1(t)dW_2(t)]\right] \\ &= \sigma_1(t)\sigma_2(t)\rho(t)\,dt \end{aligned}$$

2.1 Market with foreign currency

Example 1 can be translated into a model for a market with one risky asset and a tradable foreign currency, which is the important setting we want to discuss in this chapter.

Let $S(t)$ be the price in dollars of the risky asset.
Let $Q(t)$ denote the price (in dollars) of one unit of the foreign currency at time t. Thus $Q(t)$ is the exchange rate. Q can be thought of as a second risky asset; it fluctuates randomly and these fluctuations may be correlated with those of S (if we have reason to believe there is no correlation between S and Q we just have to set $\rho(t) = 0$ in the model below). Hence the model of Example 1 is appropriate. Following the notation in Shreve, (9.3.1)-(9.3.2), it shall be written:

$$\begin{aligned} dS(t)\,dt &= \alpha(t)S(t)\,dt + \sigma_1(t)S(t)\,dW_1(t), & (7) \\ dQ(t)\,dt &= \gamma(t)Q(t)\,dt + \sigma_2(t)Q(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]. & (8) \end{aligned}$$

Suppose that in this market one can purchase a foreign money market account earning the risk-free rate with one's foreign cash.
Let $R^f(t)$, $t \geq 0$, be the risk-free foreign rate. We will say that one unit of this money market is an account in which one unit of foreign currency is deposited at time $t = 0$ and never withdrawn.
Thus the price at time t in *foreign currency* of one foreign money market unit is

$$M^f(t) = \exp\{\int_0^t R^f(u)\,du\}.$$

The price at time t of one foreign money market unit *in dollars* is

$$N^f(t) = M^f(t)Q(t).$$

If we are investing in this market, we will certainly deposit any idle *foreign cash* in the *foreign money market*; otherwise, we forego the interest we could earn at rate R^f. Thus it is really more appropriate to write the model for the above market in terms of $S(t)$ and $N^f(t)$. An easy calculation shows that this model is:

$$\begin{aligned} dS(t)\,dt &= \alpha(t)S(t)\,dt + \sigma_1(t)S(t)\,dW_1(t), & (9) \\ dN^f(t)\,dt &= \left[\gamma(t) + R^f(t)\right] N^f(t)\,dt \\ &\quad + \sigma_2(t)N^f(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]. & (10) \end{aligned}$$

Remarks:

(i) Both equations (9), (10) are expressed in terms of dollars, not the foreign currency; and N^f is, again, the price of the foreign money market in dollars.

(ii) Both equations (9), (10) are not written in risk neutral measure setting.
(iii) Important: Even though $Q(t)$ and $N^f(t)$ are closely related, there is one crucial difference between them: under the domestic risk neutral measure $\widetilde{P}$, $D(t)Q(t)$ is **not** a martingale while $D(t)N^f(t)$ **is** a martingale. The reason is because a unit of foreign currency (without being invested into the foreign money market) loses its value over time at the rate $R^f(t)$. See also section (4.1) for more discussion.

3 A review of multi-dimensional stochastic calculus

This is more review material, collected for convenience of reference.

3.1 Multi-dimensional stochastic integration

Let $W(t) = (W_1(t), \ldots, W_d(t))$ be a d-dimensional Brownian motion, and let $\{\mathcal{F}(t);\, t \geq 0\}$ be a filtration for W. Let the vector-valued process $\Theta(t) = (\theta_1(t), \ldots, \theta(t))$ be adapted to $\{\mathcal{F}(t);\, t \geq 0\}$. For d-dimensional vectors, we use $x \cdot y = \sum_{k=1}^d x_i y_i$ to denote the inner product and $\|x\| = \sqrt{\sum_{k=1}^{d} x_k^2} = \sqrt{x \cdot x}$ to denote the norm of a vector. Accordingly, we use the following convenient notation:

$$\int_0^t \Theta(u) \cdot dW(u) = \sum_{k=1}^{d} \int_0^t \theta_k(u)\, dW_k(u).$$

For example, if we define $\underline{\sigma_i}(t) = (\sigma_{i1}(t), \ldots, \sigma_{id}(t))$, the equation for $S_i(t)$ in (4) can be written

$$dS_i(t) = \mu_i(t) S_i(t)\, dt + S_i(t) \left[\underline{\sigma_i}(t) \cdot dW(t)\right].$$

3.2 Linear stochastic differential equations

The following general fact is useful. The solution to the stochastic differential equation $dX(t) = \mu(t)X(t)\,dt + X(t)[\Theta(t)\cdot dW(t)]$ is

$$X(t) = X(0)\exp\{\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2\,du + \int_0^t \mu(u)\,du\}. \tag{11}$$

To show this expression is a solution requires just an application of the multi-dimensional Itô rule. We will not show it is the unique solution; this is done in the theory of stochastic differential equations.
A simple calculation also shows that X solves

$$dX(t) = \mu(t)X(t)\,dt + X(t)[\Theta(t)\cdot dW(t)] \tag{12}$$

if and only if

$$d\left[e^{-\int_0^t \mu(u)\,du}X(t)\right] = \left[e^{-\int_0^t \mu(u)\,du}X(t)\right][\Theta(t)\cdot dW(t)] \tag{13}$$

We will pass between these two equivalent equations frequently and without comment.

3.3 Girsanov's theorem

Let

$$Z(t)\exp\{-\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2\,du\}.$$

If it is assumed that $E[Z(T)] = 1$, then

$$\mathbf{P}^Z(A) = E[\mathbf{1}_A Z(T)], \quad A \in \mathcal{F},$$

defines a new probability measure. The multi-dimensional Girsanov theorem says that

$$W^Z(t) = W(t) + \int_0^t \Theta(u)\,du = \left(W_1(t) + \int_0^t \theta_1(u)\,du, \ldots, W_d(t) + \int_0^t \theta_d(u)\,du\right)$$

is a Brownian motion up to time T under $\mathbf{P}^Z$.

3.4 What happens when we use a general $\mathcal{F}_t^W$ martingale Z_t in the change of measure fomula

Suppose that $Z(t)$ is a $\mathcal{F}(t)$ martingale and $Z(0) = 1$. It follows that $E[Z(T)] = Z(0) = 1$. We can still define a new measure

$$\mathbf{P}^Z(A) = E[\mathbf{1}_A Z(T)], \quad A \in \mathcal{F},$$

as above (the measure $\mathbf{P}^Z$ is well-defined). However, this is a bit abstract. We did not impose any dynamics on Z_t. But we still want to learn, for example, the distribution of $W(t)$ under $\mathbf{P}^Z$. It turns out that when the filtration is generated by the Brownian motion, then the martingale representation will give us information about the dynamics of Z_t and the Girsanov's theorem will tell us about the behavior of W_t under P^Z.

(i) Martingale representation:
Assume now that the filtration $\{\mathcal{F}(t);\, t \geq 0\}$ is generated by W. Under this **important** assumption, if $Z(t)$ is a martingale with respect to $\{\mathcal{F}(t);\, t \geq 0\}$ under measure $\mathbf{P}$, then the martingale representation theorem says there exists an adapted, vector valued process $\Gamma(t) = (\gamma_1(t), \ldots, \gamma_d(t))$ such that

$$Z(t) = Z(0) + \int_0^t \Gamma(u) \cdot dW(t).$$

Suppose that $Z(0) = 1$ and that $Z(t) > 0$ for all $0 \leq t \leq T$ almost surely. By defining $\nu(t) = (\nu_1(t), \ldots, \nu_d(t)) = \dfrac{1}{Z(t)}\Gamma(t)$, one can write

$$Z(t) = 1 + \int_0^t Z(u) \frac{1}{Z(u)} \Gamma(u) \cdot dW(u) = 1 + \int_0^t Z(u)[\nu(u) \cdot dW(u)].$$

It then follows from equation (11) that

$$Z(t) = \exp\{\int_0^t \nu(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\, du\}. \tag{14}$$

(ii) Girsanov's theorem:
By applying Girsanov's theorem to this expression we obtain:

Theorem 1. *Suppose that $\{\mathcal{F}(t);\, t \geq 0\}$ is generated by W and that $Z(t)$ is an $\{\mathcal{F}(t);\, t \geq 0\}$-martingale under $\mathbf{P}$ such that $E[Z(t)] = 1$ for all t. Define $P^Z(A) = E[\mathbf{1}_A Z(T)]$, $A \in \mathcal{F}$.*

Suppose in addition that $Z(T) > 0$ almost surely. Then there is an $\{\mathcal{F}(t);\, t \geq 0\}$-adapted process $\nu(t) = (\nu_1(t), \ldots, \nu_d(t))$ so that equation (14) holds, and for this process,

$$W^Z(t) = W(t) - \int_0^t \nu(u)\, du \quad \textit{is a Brownian motion up to time } T \textit{ under } P^Z.$$

The only point we have not justified (and will not) is that if $Z(T) > 0$ almost surely, then $Z(t) > 0$ for all $t \leq T$ almost surely.
This theorem is essentially Theorem 9.2.1 in Shreve; we have just stated it more generally. It is *one of the important theorems* in this Chapter. Later on, we will replace Z_t with $D_t N_t$, where D_t is the discounted process mentioned above and N_t the actual numéraire we want to study (for example, the domestic or foreign money market). Then P^Z is the risk neutral measure associated with that numéraire. And this theorem tells us how the distribution of the Brownian motion changes under this risk neutral measure. Note that at this level, the Theorem is a bit abstract: it only tells us that the process ν exists—it does not say how to find ν. In applications, one can often determine ν from other assumptions, as we shall see in studying numéraires.

4 The domestic risk-neutral measure

Consider the model for $S(t) = (S_1(t), \ldots, S_m(t))$ given in equation (4). Assume henceforth that $\{\mathcal{F}(t);\, t \geq 0\}$ is the filtration generated by W. This allows us to employ the martingale representation theorem.
Add also to the model a risk-free rate process $R(t)$, $t \geq 0$, which is assumed to be non-negative and adapted to $\{\mathcal{F}(t);\, t \geq 0\}$. The associated discount process is denoted by $D(t) = \exp\{-\int_0^t R(u)\, du\}$.
The price of S_i, in terms of the domesitc money market, is $D_t S_i(t)$. We have the following important definition:

Definition 4.1. *The domestic risk-neutral measure for the model (4) is the probability measure $\widetilde{P}$ such that $D_t S_i(t)$ is a martingale under $\widetilde{P}$, for all i.*

The model (4) is equivalent to

$$d\,[D(t)S_i(t)] = (\mu_i(t) - R(t))D(t)S_i(t)\, dt + D(t)S_i(t) \sum_{k=1}^{d} \sigma_{ik}(t)\, dW_k(t), \qquad 1 \leq i \leq m, \tag{15}$$

as an easy calculation shows; (compare to equations (12) and (13)). We are interested in finding a domestic risk-neutral measure, assuming one exists. The essential ingredient is provided in the following theorem, which reviews material from Chapter 5 of Shreve. This review is useful because the procedure of finding the risk-neutral measure is a template for changing measure for numéraires.

Theorem 2. *Assume that there is a risk-neutral measure* $\widetilde{\mathbf{P}}$ *for model (15) given by* $\widetilde{\mathbf{P}}(A) = E[\mathbf{1}_A Z]$*, where* Z *is an* $\mathcal{F}(T)$ *measurable random variable for which* $E[Z] = 1$ *and* $\mathbf{P}(Z > 0) = 1$*. Then*

$$Z = \exp\left\{-\int_0^T \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^T \|\Theta(u)\|^2 \, du\right\}, \tag{16}$$

where $\Theta(t) = (\theta_1(t), \ldots, \theta_d(t))$ *is an* $\{\mathcal{F}(t);\, t \geq 0\}$*-adapted process that is a solution of the* market price of risk equation

$$\begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \cdots & \sigma_{1d}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) & \cdots & \sigma_{2d}(t) \\ \vdots & \vdots & & \vdots \\ \sigma_{m1}(t) & \sigma_{m2}(t) & \cdots & \sigma_{md}(t) \end{pmatrix} \cdot \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \vdots \\ \theta_d(t) \end{pmatrix} = \begin{pmatrix} \mu_1(t) - R(t) \\ \mu_2(t) - R(t) \\ \vdots \\ \mu_m(t) - R(t) \end{pmatrix}, \qquad 0 \leq t \leq T. \tag{17}$$

If this equation has a unique solution, the risk-neutral measure is unique. Under $\widetilde{\mathbf{P}}$*,*

$$\widetilde{W}(t) = \left(W_1(t) + \int_0^t \theta_1(u)\, du, \ldots, W_d(t) + \int_0^t \theta_d(u)\, du\right) \tag{18}$$

is a Brownian motion up to time t*.*

Proof: The process $Z(t) = E[Z \mid \mathcal{F}(t)]$ is a martingale and since Z is $\mathcal{F}(T)$-measurable, $Z(T) = Z$. By Theorem 1, there is an adapted process ν such that $Z(t) = \exp\{\int_0^t \nu(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2 \, du\}$. Equation (16) then follows if we set $\Theta(t) = -\nu(t)$. By Girsanov, the process $\widetilde{W}$ defined in equation (18) is a Brownian motion up to time T under $\widetilde{\mathbf{P}}$. From (18), $dW_i(t) = d\widetilde{W}_i(t) - \theta_i(t)\, dt$. By using this substitution in the equation (15) for $D(t)s_i(t)$,

$$d\left[D(t)S_i(t)\right] = \left(\mu_i(t) - R(t) - \sum_{k=1}^{d} \sigma_{ik}(t)\theta_k(t)\right) D(t)S_i(t)\, dt + D(t)S_i(t) \sum_{k=1}^{d} \sigma_{ik}(t)\, d\widetilde{W}_k(t). \tag{19}$$

This must be a martingale under the risk-neutral measure $\widetilde{\mathbf{P}}$ for all i; that is what it means for $\widetilde{\mathbf{P}}$ to be a risk-neutral measure. Thus the 'dt' term in (19) must be 0 for each i:

$$\sum_{k=1}^{d} \sigma_{ik}(t)\theta_k(t) = \mu_i(t) - R(t), \qquad 1 \leq i \leq m.$$

The matrix form of these equations is just equation (17) of the theorem statement. This completes the proof. ◇

As a consequence of the proof, the stochastic differential equation model for the discounted prices under the risk-neutral measure is

$$d\left[D(t)S_i(t)\right] = D(t)S_i(t) \sum_{k=1}^{d} \sigma_{ik}(t)\, d\widetilde{W}_k(t), \qquad 1 \leq i \leq m.$$

Remarks:

1) The equations summarized by (17) are called the *market price of risk equations.* The difference $\mu_i(t) - R(t)$ can be regarded as a risk premium; it is the amount by which the expected rate of gain of the asset is larger than the risk-free rate. Investors typically demand $\mu_i(t) - R(t)$ to be positive before investing in i, to make up for the fact that the investment carries risk. The expression $\mu_i(t) - R(t) = \sum_{k=1}^{d} \sigma_{ik}(t)\theta_k(t)$ may be thought of as a decomposition of $\mu_i(t) - R(t)$ into a sum contributions from each source of random fluctations of $S_i(t)$; $\theta_k(t)$ is effectively a price per unit of volatility of the contribution $\sigma_{ik}(t)\theta_k(t)$.

2. Theorem 2 implies that a necessary condition for the existence of a risk-neutral measure is that (17) must have a solution $\Theta(t)$. However, having a solution to (17) is not by itself a sufficient condition for the existence of a risk neutral measure. If $\Theta(t)$ is a solution and $Z = \exp\left\{-\int_0^T \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^T \|\Theta(u)\|^2\, du\right\}$, one must check in addition that $E[Z] = 1$, in order that $\widetilde{\mathbf{P}}^Z$ define a probability measure.

4.1 Model with foreign money market under the domestic risk-neutral measure

Consider the model for a risky asset, a foreign currency and a foreign money market introduced above in Section (2.1). Now add a domestic money market, with risk-free rate $R(t)$, $t \geq 0$.

Recall that the price at time t in *foreign currency* of one foreign money market unit is

$$M^f(t) = \exp\{\int_0^t R^f(u)\,du\}.$$

Given the exchange rate $Q(t)$, the price *in dollars* of a unit of the foreign money market is

$$N^f(t) = M^f(t)Q(t).$$

So there are 2 risky assets in this model:

$$\begin{aligned} dS(t)\,dt &= \alpha(t)S(t)\,dt + \sigma_1(t)S(t)\,dW_1(t), \\ dN^f(t)\,dt &= \left[\gamma(t) + R^f(t)\right] N^f(t)\,dt \\ &\quad + N^f(t)\sigma_2(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]. \end{aligned}$$

As definition (4.1) states, a domestic risk-neutral measure for this model must make $D(t)S(t)$ and $D(t)N^f(t) = D(t)M^f(t)Q(t)$ into martingales.

Equation (17) in this case is

$$\begin{pmatrix} \sigma_1(t) & 0 \\ \sigma_2(t)\rho(t) & \sigma_2(t)\sqrt{1-\rho^2(t)} \end{pmatrix} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} \alpha(t) - R(t) \\ \gamma(t) + R^f(t) - R(t) \end{pmatrix} \tag{20}$$

Assume there is a unique, risk-neutral measure for (9)–(10). By Theorem 2 equation (20) must then have a unique solution. Indeed it will, if $\sigma_1(t) > 0$, $\sigma_2(t) > 0$, and $-1 < \rho(t) < 1$ for all t with probability one. This solution is

$$\theta_1(t) = \frac{\alpha(t) - R(t)}{\sigma_1(t)}, \quad \theta_2(t) = \frac{1}{\sigma_2(t)\sqrt{1-\rho^2(t)}}\left[\gamma(t) + R^f(t) - R(t) - \sigma_2(t)\rho(t)\theta_1(t)\right].$$

Let $\widetilde{W}(t) = (W_1(t) + \int_0^t \theta_1(u)\,du, W_2(t) + \int_0^t \theta_2(u)\,du)$. Then one easily derives

$$\begin{aligned} dS(t) &= R(t)S(t)\,dt + \sigma_1(t)S(t)\,d\widetilde{W}_1(t) \\ dN^f(t) &= R(t)N^f(t)\,dt + N^f(t)\sigma_2(t)\left[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\right]. \end{aligned}$$

Remark 4.2. *This is similar to the situation in the classical Black-Scholes model, in which we consider 2 assets: the (domestic) money market and the stock S_t. The only difference is in the Black-Scholes model, the dynamics of the domestic money market does not have a Brownian motion component:*

$$dN(t) = R(t)N(t)dt.$$

Also note that the Brownian motion component of the foreign money market comes from the dynamics of the exchange rate $Q(t)$, not from the dynamics of $M^f(t)$ itself.

Remark 4.3. *We* **cannot** *require the foreign exchange rate $Q(t)$ to satisfy the condition $D(t)Q(t)$ being a martingale under the domestic risk neutral measure $\widetilde{P}$. This is because $Q(t)$ is the price of the foreign currency, which loses value over time if not invested into the foreign money market. In fact* **if** *$D(t)Q(t)$ were a martingale under the domestic risk neutral measure and the foreign interest rate is not identically 0 then we would have an arbitrage opportunity, as the following lemma shows.*

Lemma 4.4. *Suppose that $D(t)Q(t)$ is a martingale under $\widetilde{P}$ and $R^f(t)$ is not identically 0. Then an arbitrage opportunity exists.*

Proof. Consider a contract that pays 1 unit of foreign currency at time T. The value of this contract at time 0 is

$$V_0 = \tilde{E}(D(T)Q(T)).$$

By the assumption that $D(t)Q(t)$ is a martingale under $\widetilde{P}$, we have $V_0 = Q_0$. But this implies an arbitrage opportunity since at time 0, we can sell such a contract for $Q(0)$, use $Q(0)$ to buy 1 unit of foreign currency and invest in the foreign money market. At time T we would have $e^{R^f(t)dt}$ in units of foreign currency which is larger than 1 since $R^f(t)$ is not identically 0. We can then use 1 foreign currency to close out the contract and make a riskless profit.

5 Numéraires

Up to now, we have always assumed that prices were given in units of a fixed, domestic currency, which for concreteness we take to be US dollars. One could choose other units to measure prices, and it is often convenient, even necessary, to do so.

Let the price in dollars of some given asset or financial instrument be denoted $N(t)$. Let $S(t)$ be the price in dollars of any other asset. Then the ratio

$$S^{(N)} = \frac{S(t)}{N(t)}$$

is the price of the asset corresponding to S in units of the asset corresponding to N. In this situation, N is referred as the numéraire. The asset used for the numéraire could in principle be almost anything— a risky asset in the market, a foreign currency, a money market account, an index based on the market, or the price of a derivative.

Example 5.1. *Let* $R(t)$ *denote the (domestic) risk-free rate. It is common to think of* $R(t)$ *as the rate available from a money market account which can be added to or withdrawn from at will. One unit of a money market account is defined to be the value of* \$1 *invested at time* $t = 0$ *and left in the account. This value in dollars at time* t *is* $M(t) = \exp\{\int_0^t R(u)\,du\}$*. If* $S(t)$ *is the price in dollars of an asset at time* t*, its price in units of the money market is*

$$\frac{S(t)}{\exp\{\int_0^t R(u)\,du\}} = e^{-\int_0^t R(u)\,du} S(t).$$

This is just the discounted price, or present value, of $S(t)$*. So we can think of discounting as an example of pricing in money market account units.*

Example 5.2. *(Non-example) Consider the market model studied in Examples 2 and 3. This consists of an asset with price (in dollars)* $S(t)$*, an exchange rate* $Q(t)$ *(dollars per unit of foreign currency), a domestic risk-free rate* $R(t)$*, and a foreign risk-free rate* $R^f(t)$*. Let* $M(t) = \exp\{\int_0^t R(u)\,du\}$*.*
There are many choices for denominating prices. A tempting example is to use the foreign currency as the numéraire. In this case, $S^{(Q)}(t) = S(t)/Q(t)$ *is the price of the asset in units of the foreign currency, while a unit of the domestic money market in the foreign currency is* $M^{(Q)}(t) = M(t)/Q(t)$*. However, this should not be done, because under the domestic risk neutral measure,* $Q(t)$ *is NOT a martingale (See the discussion in section* (4.1) *and in section* (5.1)*). This is also consistent with our remark at the beginning of this note that we will only use non-dividend paying asset as numéraire.* $Q(t)$*, as denoting the price of the foreign currency, is a dividend paying asset with dividend rate* R^f*.*

Example 5.3. *One could also use as numéraire the value in dollars* $N^f(t) = M^f(t)Q(t)$ *of a unit of the foreign money market. Then, in this unit*

$$S^{(N^f)}(t) = \frac{S(t)}{M^f(t)Q(t)} = e^{-\int_0^t R^f(u)\,du} S^{(Q)}(t).$$

is the price of the asset, and

$$\frac{Q(t)}{M^f(t)Q(t)} = e^{-\int_0^t R^f(u)\,du}$$

is the value of a unit of foreign currency.

5.1 Change of measure for change of numéraire

Risk-neutral pricing theory should not depend on the unit of price. If there is a risk-neutral measure when the price is in dollars, then there ought to be a risk-neutral measure $\widetilde{\mathbf{P}}^N$ for pricing with respect to N, for any numéraire N. This section addresses how to find $\widetilde{\mathbf{P}}^N$.

We will always start out with a risk-neutral model for $S(t) = (S_1(t), \ldots, S_d(t))$, given on a probability space $(\Omega, \mathcal{F}, \widetilde{\mathbf{P}})$, with filtration $\{\mathcal{F}(t);\, t \geq 0\}$, and a risk-free (domestic) rate $R(t)$, $t \geq 0$. As usual, $D(t) = \exp\{-\int_0^t R(u)\,, du\}$ denotes the discount factor. Thus we can also see this as starting out with *our default probability measure* as the domestic risk neutral measure.

Let $N(t)$, $t \geq 0$, be a strictly positive, numéraire process. Since $N(t)$ represents a price and the model is risk-neutral, $D(t)N(t)$, $t \geq 0$, is a martingale. In particular,

$$\tilde{E}\left[D(T)N(T)\right] = N(0), \quad \text{for any } T \geq 0. \tag{21}$$

Let T be the time horizon for which we want to study the market. It follows that

$$\widetilde{\mathbf{P}}^{(N)}(A) = \tilde{E}\left[\mathbf{1}_A \frac{D(T)N(T)}{N(0)}\right]$$

defines a new probability measure.

Theorem 3. *$\widetilde{\mathbf{P}}^{(N)}$ is a risk-neutral measure for pricing with respect to N in the following sense: for each i, $1 \leq i \leq m$, $S_i^{(N)}(t)$ is a martingale with respect to $\{\mathcal{F}(t);\, t \geq 0\}$ up to time T under $\widetilde{\mathbf{P}}^{(N)}$.*

Proof: The proof is an application of the formula for computing conditional expectations under a change of measure. Shreve states a special case in Lemma 5.2.2. The formula implies that for any sub-σ-algebra $\mathcal{G}$,

$$\tilde{E}^{(N)}\left[X \,\middle|\, \mathcal{G}\right] = \frac{\tilde{E}\left[X\frac{D(T)N(T)}{N(0)} \,\middle|\, \mathcal{G}\right]}{\tilde{E}\left[\frac{D(T)N(T)}{N(0)} \,\middle|\, \mathcal{G}\right]}. \tag{22}$$

In this formula $\tilde{E}^{(N)}$ represents expectation with respect to $\widetilde{\mathbf{P}}^{(N)}$. Apply equation (22) with $X = S_i^{(N)}(T)$ and $\mathcal{G} = \mathcal{F}(t)$. The result is

$$\tilde{E}^{(N)}\left[S_i^{(N)}(T) \,\middle|\, \mathcal{F}(t)\right] = \frac{\tilde{E}\left[\frac{S_i(T)}{N(T)}\frac{D(T)N(T)}{N(0)} \,\middle|\, \mathcal{F}(t)\right]}{\tilde{E}\left[\frac{D(T)N(T)}{N(0)} \,\middle|\, \mathcal{F}(t)\right]} = \frac{\tilde{E}\left[D(T)S_i(T) \,\middle|\, \mathcal{F}(t)\right]}{\tilde{E}\left[D(T)N(T) \,\middle|\, \mathcal{F}(t)\right]}.$$

But $D(u)S_i(u)$ and $D(u)N(u)$ are both martingales under $\widetilde{\mathbf{P}}$, and so $\tilde{E}\left[D(T)S_i(T)\;\middle|\;\mathcal{F}(t)\right] = D(t)S_i(t)$ and $\tilde{E}\left[D(T)N(T)\;\middle|\;\mathcal{F}(t)\right] = D(t)N(t)$. Hence

$$\tilde{E}^{(N)}\left[S_i^{(N)}(T)\;\middle|\;\mathcal{F}(t)\right] = \frac{D(t)S_i(t)}{D(t)N(t)} = \frac{S_i(t)}{N(t)} = S_i^{(N)}(t).$$

This shows that $S_i^{(N)}(u)$, $0 \le u \le T$, is a martingale under $\widetilde{\mathbf{P}}^{(N)}$. ◇

Remark 5.4. *Intuitively, the foreign risk neutral measure $\widetilde{P}^{N^f}$ should satisfy the condition that the discounted (under the foreign interest rate) risky asset price quoted in the foreign currency is a martingale under $\widetilde{P^{N^f}}$. This is indeed true in our framework: Let S_t be the dynamics of the risky asset quoted in dollars. Then $\frac{S_t}{Q_t}$ is the price of the risky asset quoted in foreign currency. By our construction,*

$$D^f(t)\frac{S_t}{Q_t} = \frac{S_t}{N^f(t)} = S^{(N^f)}(t)$$

is a martingale under $\widetilde{P^{N^f}}$.

5.2 Pricing under a change of numéraire

Suppose we have a financial product that pays $V(T)$ dollars at time T. Then the (domestic) risk neutral price of this product at time t is

$$V(t) = \tilde{E}\Big[\frac{D(T)}{D(t)}V(T)|\mathcal{F}(t)\Big]$$

(since $D(t)V(t)$ is a martingale under $\tilde{P}$).
What is the corresponding pricing formula when $V(t)$ is denoted under the unit of a numéraire $N(t)$? Arguing similar to the proof of Theorem (3) we will see that

$$\frac{V(t)}{N(t)} = \tilde{E}^{(N)}\Big[\frac{V(T)}{N(T)}|\mathcal{F}(t)\Big].$$

Thus denoting $V^{(N)}(t) := \frac{V(t)}{N(t)}$ we have

$$V^{(N)}(t) = \tilde{E}^{(N)}\Big[V^{(N)}(T)|\mathcal{F}(t)\Big].$$

This equation is meaningful by itself. It says that the price in the unit of numéraire $N(t)$ of the financial product is the conditional expectation under the corresponding

risk neutral measure of the terminal value, also expressed in the same unit of numéraire. Note that the domestic risk neutral pricing formula is a special case of this when we use $N(t) = \frac{1}{D(t)}$, the domestic money market account.
It is also important to remember that here $V(t)$ is again in dollars, or the domestic currency, and $N(t)$ is the price in dollars of the numéraire of interest. To see a consequence of this, see the below section on pricing a financial product quoted in foreign currency.

5.3 Effect of change of numéraire

In section V, no assumptions were made on the nature of the price model. In this section, we specialize to the multi-asset model stated above and written under the risk-neutral measure as

$$d\left[D(t)S_i(t)\right] = D(t)S_i(t)\sum_{k=1}^{d}\sigma_{ik}(t)\,d\widetilde{W}_k(t), \qquad 1 \le i \le m. \tag{23}$$

In addition, we impose the assumption that $\{\mathcal{F}(t);\, t \ge 0\}$ is generated by $\widetilde{W}$.
Let N be a numéraire process and $\widetilde{\mathbf{P}}^{(N)}$ the risk-neutral measure for N. $\widetilde{W}$ is no longer a Brownian motion under $\widetilde{\mathbf{P}}^{(N)}$. The object of this section is use Theorem 1 and Girsanov's theorem to identify an appropriate Brownian motion $\widetilde{W}^{(N)}(t)$ under $\widetilde{\mathbf{P}}^{(N)}$ and to rewrite equation (23) using it.
Now $D(t)N(t)$ is a martingale under $\widetilde{\mathbf{P}}$, and $N(t)$, as a numéraire, is strictly positive for all t. So Theorem 1 implies there is a process ν so that

$$\frac{D(t)N(t)}{N(0)} = \exp\{\int_0^t \nu(u)\cdot d\widetilde{W}(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\,du\} \tag{24}$$

Since $\widetilde{\mathbf{P}}^{(N)}(A) = \tilde{E}[\mathbf{1}_A\frac{D(T)N(T)}{N(0)}]$, it also follows from Theorem 1 that $\widetilde{W}^{(n)}(t) = \widetilde{W}(t) - \int_0^t \nu(u)\,du$ is a Brownian motion under $\widetilde{\mathbf{P}}^{(N)}$ up to time T.
Define $\sigma_i(t) = (\sigma_{i1}(t), \ldots, \sigma_{id}(t))$, so that (23) may be written compactly as

$$dD(t)S_i(t) = D(t)S_i(t)\left[\sigma_i(t)\cdot d\widetilde{W}(t)\right].$$

By (11), the solution to this equation is

$$D(t)S_i(t) = S_i(0)\exp\{\int_0^t \sigma_i(u)\cdot d\widetilde{W}(u) - \frac{1}{2}\int_0^t \|\sigma_i(u)\|^2\,du\}.$$

Thus, using the representation (24) for $D(t)N(t)$,

$$\begin{aligned} S_i^{(N)}(t) &= \frac{S_i(t)}{N(t)} = \frac{D(t)S_i(t)}{D(t)N(t)} \\ &= \frac{S_i(0)}{N(0)} \exp\{\int_0^t [\sigma_i(u) - \nu(u)] \cdot d\widetilde{W}(u) - \frac{1}{2}\int_0^t (\|\sigma_i(u)\|^2 - \|\nu(u)\|^2)\, du\} \end{aligned}$$

Replace $d\widetilde{W}$ in this expression by $d\widetilde{W}^{(N)}(t) + \nu(t)\, dt$. Note first that

$$\begin{aligned} &\exp\{\int_0^t [\sigma_i(u) - \nu(u)] \cdot [d\widetilde{W}^{(N)}(u) + \nu(u)\, du] \\ &= \int_0^t [\sigma_i(u) - \nu(u)] \cdot d\widetilde{W}^{(N)}(u) + \int_0^t \sigma_i(u) \cdot \nu(u)\, du - \int_0^t \nu(u) \cdot \nu(u)\, du \\ &= \int_0^t [\sigma_i(u) - \nu(u)] \cdot d\widetilde{W}^{(N)}(u) + \int_0^t \sigma_i(u) \cdot \nu(u)\, du - \int_0^t \|\nu(u)\|^2\, du \end{aligned}$$

It follows that

$$\begin{aligned} &S_i^{(N)}(t) \\ &= \frac{S_i(0)}{N(0)} \exp\{\int_0^t [\sigma_i(u) - \nu(u)] \cdot d\widetilde{W}^{(N)}(u) - \frac{1}{2}\int_0^t (\|\sigma_i(u)\|^2 - 2\underline{\sigma}_i(u) \cdot \nu(u) + \|\nu(u)\|^2)\, du\}. \end{aligned}$$

But

$$\begin{aligned} \|\underline{\sigma}_i(u) - \nu(u)\|^2 &= [\sigma_i(u) - \nu(u)] \cdot [\sigma_i(u) - \nu(u)] \\ &= \|\sigma_i(u)\|^2 - 2\sigma_i(u) \cdot \nu(u) + \|\nu(u)\|^2) \end{aligned}$$

Thus

$$S_i^{(N)}(t) = \frac{S_i(0)}{N(0)} \exp\{\int_0^t [\sigma_i(u) - \nu(u)] \cdot d\widetilde{W}^{(N)}(u) - \frac{1}{2}\int_0^t \|\sigma_i(u) - \nu(u)\|^2\, du\}$$

It follows from equation (11) that

$$dS_i^{(N)}(t) = S_i^{(N)}(t)[\sigma_i(t) - \nu(t)] \cdot d\widetilde{W}^{(N)}(t) = S_i^{(N)}(t) \sum_{k=1}^{d} (\sigma_{ik}(t) - \nu_k(t))\, d\widetilde{W}_k^{(N)}(t) \quad (25)$$

This is an interesting equation because it shows exactly how the volatility of $S_i^{(N)}$ differs from that of S_i. Of course, we expect them to differ because N itself has volatility and $S_i^{(N)}(t) = S_i(t)/N(t)$. In fact, from the expression (24) and from (11) one finds that

$$dN(t) = R(t)N(t)\, dt + \sum_{k=1}^{d} \nu_k(t)\, d\widetilde{W}_k(t),$$

so $\nu_k(t)$ is the component of the volatility of N at time t due to $\widetilde{W}_k$.

6 Foreign risk-neutral measure

The discussion of the previous section established the existence of ν, but not a formula for it. In examples it can be found explicitly if the numéraire is defined explicitly.
Consider the example of a market with an asset and foreign currency formulated above. Its risk neutral version was derived in Example 3 and is

$$\begin{aligned} dS(t) &= R(t)S(t)\,dt + \sigma_1(t)S(t)\,d\widetilde{W}_1(t) && (26)\\ dN^f(t) &= R(t)N^f(t)\,dt + N^f(t)\sigma_2(t)\left[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\right]. && (27)\end{aligned}$$

Recall that $N^f(t) = \exp\{\int_0^t R^f(u)\,du\}Q(t)$ is the dollar value of one unit of **the foreign money market account**. *We shall use it as the numéraire in this section.* The domestic discount factor is $D(t) = \exp\{-\int_0^t R(u)\,du\}$.
From (27),

$$\begin{aligned} d[D(t)N^f(t)] &= D(t)N^f(t)\sigma_2(t)\left[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\right]\\ &= D(t)N^f(t)\left(\sigma_2(t)\rho(t), \sigma_2(t)\sqrt{1-\rho^2(t)}\right)\cdot d\widetilde{W}(t).\end{aligned}$$

Note: $D(0)N^f(0) = Q(0)$. It follows from (11) that

$$D(t)N^f(t) = Q(0)\exp\{\int_0^t \nu(u)\cdot d\widetilde{W}(t) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\,du\},$$

where $\nu(t) = \left(\sigma_2(t)\rho(t), \sigma_2(t)\sqrt{1-\rho^2(t)}\right)$.
The risk-neutral measure for denominating prices in units of the foreign money market up to time T, or *the foreign risk neutral measure* in short, is therefore defined by

$$\widetilde{\mathbf{P}}^{(N^f)}(A) = \tilde{E}\left[\mathbf{1}_A\frac{D(T)N^f(T)}{Q(0)}\right] = \tilde{E}\left[\mathbf{1}_A\exp\{\int_0^t \nu(u)\cdot d\widetilde{W}(t) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\,du\}\right],$$

and

$$\widetilde{W}^{(N^f)}(t) = \left(\widetilde{W}_1(t) - \int_0^t \sigma_2(u)\rho(u)\,du,\ \widetilde{W}_2(t) - \int_0^t \sigma_2(u)\sqrt{1-\rho^2(u)}\,du\right)$$

is a Brownian motion up to time T under $\widetilde{\mathbf{P}}^{(N^f)}$.
The price of the asset with respect to numéraire $N^f(t)$ is $S^{(N^f)}(t) = S(t)/N^f(t)$. It is the *present value* of the asset price in units of the foreign currency (or just simply

the value of the asset price at time t in units of the foreign money market). It is a martingale with respect to $\widetilde{\mathbf{P}}^{(N^f)}$. By applying (25),

$$dS^{(N^f)}(t) = S^{(N^f)}(t)\left[(\sigma_1(t) - \sigma_2(t)\rho(t))\, d\widetilde{W}_1^{(N^f)}(t) - \sigma_2(t)\sqrt{1-\rho^2(t)}\, d\widetilde{W}_2^{(N^f)}(t)\right], \quad t \leq T.$$

Because $\widetilde{W}_2$ contributes to the volatility of $N^f(t)$, $d\widetilde{W}_2^{(N^f)}(t)$ contributes to the volatility of $S^{(N^f)}(t)$.

6.1 Pricing a financial product quoted in foreign currency

Suppose we have a financial product that pays $V(T) := \Phi(S\cdot, Q\cdot)$ *units of foreign currency* at time T. Then we have the following lemma

Lemma 6.1. *The risk neutral price (in foreign currency) of the above product is*

$$V(t) = \tilde{E}^{(N^f)}\Big(e^{-\int_t^T R^f(u)du} V(T) \Big| \mathcal{F}(t)\Big).$$

Note that the result is very intuitive: to price a financial product quoted in foreign currency, we take conditional expectation under the foreign risk neutral measure, discounted under the foreign interest rate.
Proof. The proof of this Lemma relies on the result of Section (5.2). Using the foreign money market $N^f(t)$ as numéraire, the value of $V(T)$ denoted in the units of $N^f(T)$ is

$$\frac{V(T)Q(T)}{N^f(T)}.$$

To price $V(T)$ using $N^f(T)$ as numéraire we need to use $\widetilde{P}^{(N^f)}$. Thus we have

$$V^{(N^f)(t)} = \tilde{E}^{(N^f)}\Big[\frac{V(T)Q(T)}{N^f(T)}\Big|\mathcal{F}(t)\Big].$$

Note that $N^f(T) = M^f(T)Q(T)$, and $V^{(N^f)(t)} = \frac{V(t)}{M^f(t)}$ (be careful to distinguish $V(t)$ here and the $V(t)$ in Section (5.2). The $V(t)$ here is the risk neutral price in foreign currency. The $V(t)$ in Section (5.2) is the risk neutral price in domestic currency. They are unrelated).
After simplifying, we get

$$\frac{V(t)}{M^f(t)} = \tilde{E}^{(N^f)}\Big[\frac{V(T)}{M^f(T)}\Big|\mathcal{F}(t)\Big].$$

Since $M^f(t) = e^{\int_0^t R^f(u)du}$ the conclusion follows.

Remark: Alternatively, the risk neutral price (in dollars) of this financial product is

$$\tilde{V}_t = \tilde{E}\Big[\frac{D(T)V(T)Q(T)}{D(t)}\Big|\mathcal{F}(t)\Big].$$

But we have

$$\begin{aligned}\tilde{E}\Big[\frac{D(T)V(T)Q(T)}{D(t)}\Big|\mathcal{F}(t)\Big] &= \tilde{E}^{(N^f)}\Big[\frac{D(T)V(T)Q(T)}{D(t)}\frac{Q(t)M^f(t)D(t)}{Q(T)M^f(T)D(T)}\Big|\mathcal{F}(t)\Big] \\ &= \tilde{E}^{(N^f)}\Big[\frac{D^f(T)V(T)Q(t)}{D^f(t)}\Big|\mathcal{F}(t)\Big].\end{aligned}$$

Thus dividing by $Q(t)$ on both sides gives

$$D^f(t)V(t) = \tilde{E}^{(N^f)}\Big[D^f(T)V(T)\Big|\mathcal{F}(t)\Big].$$

7 The exchange rate

Recall the exchange rate model. There is asset price $S(t)$, foreign exchange rate $Q(t)$, domestic money market rate $R(t)$ and foreign money market rate $R^f(t)$. Recall

$$N^f(t) = \exp\{\int_0^t R^f(u)\,du\}Q(t) \tag{28}$$

is the dollar value of one unit of the foreign money market account. The risk-neutral model when prices are in dollars is

$$\begin{aligned} dS(t) &= R(t)S(t)\,dt + \sigma_1(t)S(t)\,d\widetilde{W}_1(t) \\ dN^f(t) &= R(t)N^f(t)\,dt + N^f(t)\sigma_2(t)\Big[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\Big] \end{aligned} \tag{29}$$

The risk-neutral measure $\widetilde{\mathbf{P}}^{(N^f)}$ when prices are denominated using N^f as numéraire is given in Shreve, page 386, equation (9.3.17) and in the previous set of lecture notes. Here, we make some remarks concerning the exchange rate process $Q(t)$.

7.1 The exchange rate under the domestic risk-neutral measure

It follows from equations (28), (29) that

$$dQ(t) = [R(t) - R_f(t)]\,Q(t)\,dt + Q(t)\sigma_2(t)\Big[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\Big] \tag{30}$$

When dealing with Q alone it is convenient to write this in a simpler form. Define

$$\widetilde{W}_3(t) = \int_0^t \left[\rho(t)\, d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\, d\widetilde{W}_2(t)\right].$$

Observe that

$$\begin{aligned}[d\widetilde{W}_3(t)]^2 &= \left[\rho(t)\, d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\, d\widetilde{W}_2(t)\right]^2 \\ &= \rho^2(t)\, dt + (1-\rho^2(t))\, dt = dt.\end{aligned}$$

Then by Itô's rule,

$$\widetilde{W}_3^2(t) = \int_0^t \widetilde{W}_3(u) \left[\rho(t)\, d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\, d\widetilde{W}_2(t)\right] + t.$$

Hence $\widetilde{W}_3^2(t) - t$ is a martingale. $\widetilde{W}_3(t)$ is also a continuous martingale starting at 0, and so Lévy's theorem implies that $\widetilde{W}_3(t)$ is itself a Brownian motion. Using $\widetilde{W}_3$,

$$dQ(t) = [R(t) - R_f(t)]\, Q(t)\, dt + \sigma_2(t) Q(t)\, d\widetilde{W}_3(t). \tag{31}$$

Remark: The foreign exchange rate behaves exactly like a risky asset that pays dividents at rate $R^f(t)$. Equation (31) is the same as equation (5.5.6) in Shreve for a dividend-paying asset if $A(t)$ in that equation is replaced by $R^f(t)$.

7.2 Black-Scholes formula for a Call option on the exchange rate

Let $\sigma_2(t) = \sigma_2$ be constant, and also let $R(t) = r$ and $R^f(t) = r^f$ be constant. Then equation (31) becomes

$$dQ(t) = \left[r - r^f\right] Q(t)\, dt + \sigma_2 Q(t)\, d\widetilde{W}_3(t). \tag{32}$$

The solution to this equation is

$$Q(t) \quad = \quad Q(0) \exp\{\sigma_2 \widetilde{W}_3(t) + (r - r^f - \frac{1}{2}\sigma^2)t\}. \tag{33}$$

We can look at $Q(t)$ (*from a computational point of view*) as the Black-Scholes price of an asset following the geometric Brownian motion model, when the volatility is σ_2 and *the risk free rate is* $r^f - r$.

The fact that $Q(t)$ is a classical Black-Scholes price gives immediate formulas for options on the exchange rate in the constant coefficient case, which we will develop below.
Suppose that the risk free rate is r and under $\tilde{P}$, a stock S_t has dynamics:

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t.$$

Let $C(T-t, x, K, r, \sigma)$ the price of at time t of a European call on S with strike K, conditioned on $S_t = x$. That is

$$C(T-t, x, K, r, \sigma) = \tilde{E}\Big(e^{-r(T-t)}(S_T - K)^+ \Big| S_t = x\Big).$$

Then the Black-Scholes formula for $C(T-t, x, K, r, \sigma)$ is

$$\begin{aligned} C(T-t, x, K, r, \sigma) &= e^{-r(T-t)}\tilde{E}\left[\left(xe^{\sigma\widetilde{W}(T-t))+(r-\sigma^2/2)(T-t)} - K\right)^+\right] \\ &= xN\left(\frac{\ln(x/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad - Ke^{-r(T-t)}N\left(\frac{\ln(x/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

Consider now a European call option on $Q(T)$ at strike K, for the model of (33). This can also be looked at as a call option with strike K on *a unit of foreign currency, quoted in domestic currency.*
According to risk-neutral pricing, the value of this option at time t is

$$\begin{aligned} V(t) &= e^{-r(T-t)}\tilde{E}\left[(Q(T) - K)^+ \mid \mathcal{F}(t)\right] \\ &= e^{-r^f(T-t)}e^{-(r-r^f)(T-t)}\tilde{E}\left[(Q(T) - K)^+ \mid Q(t)\right]. \end{aligned}$$

But evaluating $e^{-(r-r^f)(T-t)}\tilde{E}\left[(Q(T) - K)^+ \mid Q(t)\right]$ is exactly the same as evaluating the price of a European call when the risk free rate is $r - r^f$ and the volatility is σ_2. Therefore,

$$V(t) = e^{-r^f(T-t)}C(T-t, Q(t), K, r - r^f, \sigma_2).$$

This is called the Garman-Kohlhagen formula. You can also recover this formula from the formula (5.5.12) in Shreve for the price of a call on dividend-paying asset. Just replace a in this formula by r^f.

7.3 The exchange rate from the foreign currency viewpoint

Starting with the model (28)-(29), suppose we use the foreign currency money market $N^f(t)$ as the numéraire. In the previous lecture we found that

$$\widetilde{W}^{(N^f)}(t) = (\widetilde{W}_1(t) - \int_0^t \sigma_2(u)\rho(u)\,du, \widetilde{W}_2(t) - \int_0^t \sigma_2(u)\sqrt{1-\rho^2(u)}\,du)$$

is a Brownian motion under $\widetilde{\mathbf{P}}^{(N^f)}$ and we showed

$$dS^{(N^f)}(t) = S^{(N^f)}(t)\left[(\sigma_1(t) - \sigma_2(u)\rho(u))\,d\widetilde{W}_1^{(N^f)}(t) - \sigma_2(t)\sqrt{1-\rho^2(t)}\,d\widetilde{W}_2^{(N^f)}(t)\right]$$

To completely describe the model from the viewpoint of the foreign currency we should also look at the dollar to foreign currency exchange rate $1/Q(t)$, which is the value of one dollar in units of the foreign currency. The equation for this should have a form symmetrical to the equation (32) for $Q(t)$ when units are in dollars. Indeed,

$$d\left[\frac{1}{Q(t)}\right] = [R^f(t) - R(t)]\frac{1}{Q(t)}\,dt - \sigma_2(t)\frac{1}{Q(t)}\left[\rho(u)\,d\widetilde{W}_1^{(N^f)}(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2^{(N^f)}(t)\right]. \tag{34}$$

This may be verified from Itô's rule, but one can see why it must be correct by the following reasoning. From the perspective of numéraire N^f, $R^f(t)$ is the domestic risk free rate and $R(t)$ is the domestic rate, so, where $R(t) - R^f(t)$ appears in (31), $R^f(t) - R(t)$ appears in (34). The volatility terms are essentially the same because the same stochastic fluctuation is obviously driving both $Q(t)$ and $1/Q(t)$. To explain why σ_2 appears in (31) but $-\sigma_2$ appears in (34) just note that $1/Q(t)$ goes down when Q goes up and vice-versa.
Concerning this topic, the student should read section 9.3.4 on Siegel's paradox (which is not really a paradox, but arises from a misunderstanding of the correct numéraire to use in interpreting a model.)

8 Zero coupon bonds as numéraire

In this section we assume given a risk-neutral model with a stochastic interest rate process $R(t)$, $t \geq 0$.

8.1 Zero-coupon bonds

Bonds are financial instruments that promise fixed payoffs. Most bonds provide periodic payments called coupons and then a final payment consisting of a coupon and a lump sum called the *principal* or *face value*. A zero-coupon bond pays out only at the terminal time. We let $B(t,T)$ denote the price at time $t \leq T$ of a zero-coupon bond that pays \$1 at time T.
Given a risk-neutral model defined by a probability measure $\widetilde{\mathbf{P}}$, the no-arbitrage principle demands that $D(t)B(t,T)$ be a martingale in t up to time T. Since $B(T,T)=1$, it follows that

$$B(t,T) = \frac{\tilde{E}[D(T)B(T,T) \mid \mathcal{F}(t)]}{D(t)} = \frac{\tilde{E}[D(T) \mid \mathcal{F}(t)]}{D(t)} = \frac{\tilde{E}\left[e^{-\int_0^T R(u)\,du} \;\middle|\; \mathcal{F}(t)\right]}{e^{\int_0^t R(u)\,du}}. \tag{35}$$

Hence,

$$B(t,T) = \tilde{E}\left[e^{-\int_t^T R(u)\,du} \;\middle|\; \mathcal{F}(t)\right] \tag{36}$$

This is an interesting formula. If $R(\cdot)$ is a random process, and we are at time t, we do not know what R will be exactly after time t. On the other hand, the market tells us what all zero-coupon bond prices are. Any model we create for R must be consistent with (in quant lingo, must be calibrated to) the zero-coupon bond prices via (36).

8.2 Forward prices

Suppose at time t, where $t < T$, Alice contracts to buy a unit of an asset from Bob at price F at time T. This is called a forward contract. No money changes hands at time t. Let $S(u)$ denote the price of the asset as a function of time u. From Alice's perspective she is getting an option that pays off $S(T) - F$, because she is purchasing something worth $S(T)$ dollars for F dollars at time T. The value of this option at t is $D^{-1}(t)\tilde{E}[D(T)(S(T) - F) \mid \mathcal{F}(t)] = S(t) - FB(t,T)$; remember, $D(t)S(t)$ is a martingale with respect to $\widetilde{\mathbf{P}}$! If she is paying or receiving no money for the contract at time t this value should be zero. Hence

$$F = \frac{S(t)}{B(t,T)}$$

is the fair price for this contract. It is called the T-forward price and denoted by $\mathrm{For}_S(t,T)$. Really, it is the price of $S(t)$ obtained using $B(t,T)$ as a numéraire.

A trivial but important observation is that the forward price and the market price concur at time T:

$$\text{For}_S(T,T) = \frac{S(T)}{B(T,T)} = S(T). \tag{37}$$

8.3 The risk-neutral measure associated with the zero-coupon bond

Under the domestic risk neutral measure $\tilde{P}$, $D_t B(t,T)$ is a martingale. Therefore, $B(t,T)$ can be used as a numéraire. Indeed, the risk-neutral measure corresponding to numéraire $B(t,T)$, according to Theorem 3 of Lecture 9, is

$$\widetilde{\mathbf{P}}^T(A) = \tilde{E}[\mathbf{1}_A \frac{D(T)B(T,T)}{B(0,T)}] = \frac{1}{B(0,T)} \tilde{E}[\mathbf{1}_A D(T)]$$

We will call P^T, following Shreve (Definition 9.4.1), *the T-forward measure.* Consider the special case in which the filtration in the risk neutral market is generated by a single Brownian motion $\widetilde{W}$. Then in this case we know from Theorem 9.1 of Shreve that there is a process $\nu_T(u)$ such that

$$\frac{D(T)}{B(0,T)} = e^{\int_0^T \nu_T(u)\, d\widetilde{W}(u) - \frac{1}{2}\int_0^T \nu_T^2(u)\, du}$$

and that $\widetilde{W}^T(t) = \widetilde{W}(t) - \int_0^t \nu_t(u)\, du$ is a Brownian motion under $\widetilde{\mathbf{P}}^T$. (In Shreve, 9.4.2, the notation $-\sigma^*(t,T)$ stands for our $\nu_T(t)$.

8.4 Pricing under the T-forward measure

8.4.1 Pricing under the domestic risk neutral measure with random interest rate

Suppose the interest rate is R_t, an adapted process. Then the risk neutral price V_t of a Euro style financial product that pays V_T at time T is

$$V_t = \tilde{E}\Big(e^{-\int_t^T R_u du} V_T \Big| \mathcal{F}_t\Big).$$

Since all we know about R_t is that it is an adapted process, we cannot go further with this pricing formula, unless we make some assumption on R_t (which is about modeling the interest rate, the topic of next Chapter). This is certainly a complex topic. Moreover, even if we have a model for R_t, it doesn't mean the pricing formula will be simple, if $\int_t^T R_u du$ has non zero correlation with V_T, for example. However, a nice observation here is that we do not have to compute this equation under $\tilde{E}$. Indeed, recall from the section 5.2 result, we have:

$$V_t^T = \tilde{E}^T\left(V_T^T \middle| \mathcal{F}_t\right).$$

where $V_t^T := \frac{V_t}{B(t,T)}$ is the price of the product denoted in the unit of zero-coupon bond. Note that since $B(T,T) = 1$, we have $V_T^T = V_T$.
The nice thing about the pricing formula under $\tilde{P}^T$ is that it is only a conditional expectation of the terminal value, not involving other quantities like the interest rate (this is not a pure gain, since the interest rate was absorbed into $\tilde{E}^T$).
However, this suggests a new approach to the entire problem: we may directly model the assets under $\tilde{P}^T$, rather than under domestic measure $\tilde{P}$. Note that if we model the asset under $\tilde{P}^T$, then the unit of denomination (or the numéraire) is the price of zero coupon bond $B(t,T)$. In particular, if our objective is to model the stock price S_t (under $\tilde{P}$) then under $\tilde{P}^T$, we model

$$S_t^T := \frac{S_t}{B(t,T)} = For_S(t,T).$$

Pricing a call option on $S(t)$ under the domestic risk neutral measure is equivalent to pricing a call option on the forward price $For_S(t,T)$ under the T-forward measure. The advantage here is again about modeling. If we model under $\tilde{P}$ then necessarily we need to involve the model of R_t and need to know how to handle the expectation $\tilde{E}\left(e^{-\int_t^T R_u du} V_T \middle| \mathcal{F}_t\right)$. If we model under $\tilde{P}^T$ then we only need to model the forward price of S_t (which potentially maybe easier to calibrate to market parameters than modeling R_t) and then the expectation $\tilde{E}^T\left(V_T^T \middle| \mathcal{F}_t\right)$ is straight forward. The detailed computation is done in the following section.

8.4.2 Pricing a Call option under the T-forward measure

Here one assumes that the forward price of asset S, is given by the simple formula

$$d\mathrm{For}_S(t,T) = \sigma \mathrm{For}_S(t,T)\, d\widetilde{W}^T(t), \qquad t \leq T.$$

The point is that this is the Black-Scholes price model with $r = 0$, and if one looked at $S(t)$ under the original risk-neutral measure, it would not follow a Black-Scholes model with constant volatility. However it is possible to explicitly price a call.

Indeed, let $C(T-t, x, K, r, \sigma)$ denote the Black-Scholes price of a European call when the price is x, the risk-free interest rate is r and the volatility is σ. Let $V(t)$ be the dollar price of the call. Then its forward price is $V^T(t) = V(t)/B(t,T)$. But, recalling from (37) that $\mathrm{For}_S(T,T) = S(T)$, we know from risk-neutral pricing that

$$\begin{aligned} V^T(t) &= \tilde{E}^T \left[\frac{(S(T)-K)^+}{B(T,T)} \,\Big|\, \mathcal{F}(t) \right] = \tilde{E}^T \left[(\mathrm{For}_S(T,T)-K)^+ \,\Big|\, \mathcal{F}(t) \right] \\ &= \tilde{E}^T \left[(\mathrm{For}_S(T,T)-K)^+ \,\Big|\, \mathrm{For}_S(t,T) \right]. \end{aligned}$$

But since $\mathrm{For}_S(t,T)$ follows the Black-Scholes price model with $r=0$ and volatility σ,

$$V^T(t) = C(T-t, \mathrm{For}_S(t,T), K, 0, \sigma).$$

Hence,

$$V(t) = B(t,T)C(T-t, \mathrm{For}_S(t,T), K, 0, \sigma).$$

By substitution into the explicit formula for C (given above on page 2),

$$\begin{aligned} V(t) &= B(t,T)\mathrm{For}_S(t,T) N\left(\frac{\ln(\frac{\mathrm{For}_S(t,T)}{K}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \right) \\ &\quad - KB(t,T)N\left(\frac{\ln(\frac{\mathrm{For}_S(t,T)}{K}) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \right) \end{aligned}$$

This is essentially formula (9.4.9) in Shreve.
Clearly, this procedure could be applied to other cases where explicit pricing formulae are known for the Black-Scholes price model.