Asian option

Math 622

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1 Preliminary discussion

Let S_t satisfies

$$\begin{aligned} dS_t &= rS_t dt + \sigma(t, S_t) S_t dW_t \\ S(0) &= x > 0. \end{aligned}$$

Denote for $t \in [0, T]$

$$\begin{aligned} Y_t &= \int_0^t S(u) du; \\ S_{ave}(t) &= \frac{Y_t}{t}. \end{aligned}$$

Consider the Generalized Asian Option:

$$V_T = G\big(S(T), S_{ave}(T)\big).$$

Depends on the specific form G takes, we have the following types of Asian options:

(i) *Average price call*: $G(x, y) = (y - K)^+$;

(ii) *Average price put*: $G(x, y) = (K - y)^+$;

(iii) *Average strike call*: $G(x, y) = (x - y)^+$;

(iv) *Average strike put*: $G(x, y) = (y - x)^+$.

By risk neutral pricing

$$\begin{aligned} V_t &= E\Big\{e^{-r(T-t)} G\big(S_T, S_{ave}(T)\big) \Big| \mathcal{F}(t)\Big\} \\ &= E\Big\{e^{-r(T-t)} G\big(S_T, \frac{Y_T}{T}\big) \Big| \mathcal{F}(t)\Big\}. \end{aligned}$$

In the Lookback Option, we have discussed that the process (S_t, Y_t) (where $Y_t := \max_{[0,t]} S_u$) has Markov property. The same principle applies here

Principle: If $S_t, t \geq 0$ is Markov with respect to $\mathcal{F}(t)$ and $Y_t = \int_0^t S(u)du$ then $\{Y_t, S_t\}$ is also Markov with respect to $\mathcal{F}(t)$.

The intuitve reason why this principle is true is because we can write

$$Y(T) = Y_t + \int_t^T S(u)du.$$

Therefore, intuitively, to compute the conditional expectation of $Y(T)$ on $\mathcal{F}(t)$, we only need the value of Y_t plus the conditional expectation of $S(u)$ given $\mathcal{F}(t)$, which also only depends on S_t by the assumption on Markov property of S. In other words, the conditional expectation of $Y(T)$ on $\mathcal{F}(t)$ only depends on S_t, Y_t, thus the process S_t, Y_t is Markov.
Note here however that Y_t by itself is generally NOT a Markov process (same as the conclusion we draw for the running max of S_t in the Lookback option).
Thus there exists $v(t, x, y)$ such that

$$V(t) = v(t, S_t, Y_t)$$

where

$$\begin{aligned} v(t, x, y) &= E\Big\{e^{-r(T-t)} G\big(S_T, \frac{Y_T}{T}\big)\Big|S_t = x, Y_t = y\Big\} \\ &= E\Big\{e^{-r(T-t)} G\big(S_T, \frac{y + \int_t^T S_u du}{T}\big)\Big|S_t = x\Big\} \end{aligned}$$

To be able to proceed, one would need the knowledge of the joint distribution between $\int_t^T \sigma(u, S_u)dWu$ and $\int_t^T S_u du$. So without further assumption on S, this is the ultimate simplification that can be achieved to represent the option price as conditional expectation.

Remark 1.1. *If G is a linear function in x, y, however, then we can write down an explicit formula for $v(t, x, y)$. Do you see why? (Say for example, $G(x, y) = y - x$).*

2 PDE for Asian options

An explicit formula for the price in terms of expectation for an option of Asian type is not known, even if S follows the standard Black-Scholes (that is σ is a constant) model. To get an idea, observe that one will need to figure out the joint distribution of $(\int_0^t e^{W_u} du, e^{W_t})$ to start investigating how to express $v(t, x, y)$ above as an integral. Therefore, it is important do drive a PDE with boundary conditions for $v(t, x, y)$ defined above.

Note that here, unlike the case of Lookback option, the process Y_t is an *absolutely continuous* process. Indeed, its dynamics is

$$
\begin{aligned}
dY_t &= S_t dt \\
Y(0) &= 0.
\end{aligned}
$$

Therefore, applying Ito's formula and setting the "dt" term to 0 is no problem:

$$
de^{-rt}v(t, S_t, Y_t) = e^{-rt}\mathcal{L}v(t, S_t, Y_t)dt + e^{-rt}v_x(t, S_t, Y_t)S_t\sigma(t, S_t)dWt,
$$

where

$$
\mathcal{L}v(t, x, y) := -rv(t, x, y) + v_t(t, x, y) + v_x(t, x, y)rx + v_y(t, x, y)x + \frac{1}{2}v_{xx}(t, x, y)\sigma^2(t, x)x^2.
$$

Recall that from the discussion on the quadratic variation and covariation in Lecture 6a, since Y_t is a function of bounded variation, $\langle Y\rangle_t = 0$ and $\langle S, Y\rangle_t = 0$. Thus since $e^{-rt}v(t, S_t, Y_t)$ is a martingale, we set the dt term to 0 and get

$$
\begin{aligned}
&-rv(t, x, y) + v_t(t, x, y) + v_x(t, x, y)rx + v_y(t, x, y)x + \frac{1}{2}v_{xx}(t, x, y)\sigma^2(t, x)x^2 = 0, \\
&0 < x, y < \infty, 0 \le t < T.
\end{aligned}
$$

But we also need to impose boundary conditions.

(i) At $t = T$ this is clear:

$$
v(T, x, y) = G(x, \frac{y}{T}). \tag{1}
$$

(ii) At $x = 0$: when the stock price hits 0, it stays there: $S(u) = 0, u \ge t$, so $Y(u)$ remains a constant on $[t, T]$ as well. Thus

$$
\begin{aligned}
v(t, 0, Y_t) &= E(e^{-r(T-t)}G(S(T), \frac{Y_T}{T})|\mathcal{F}(t)) \\
&= E(e^{-r(T-t)}G(0, \frac{Y_t}{T})|\mathcal{F}(t)) = e^{-r(T-t)}G(0, \frac{Y_t}{T}).
\end{aligned}
$$

This implies that

$$
v(t, 0, y) = e^{-r(T-t)}G(0, \frac{y}{T}). \tag{2}
$$

(iii) It's now natural to finish with the boundary condition at $y = 0$. However, note that

$$v(t, x, 0) \quad = \quad E(e^{-r(T-t)} G(S(T), \frac{\int_t^T S_u du}{T}) | S_t = x),$$

but in general we don't know what this is.

(iv) We instead then tries to seek the "boundary condition" for y at ∞. Suppose that

$$\lim_{y \to \infty} G(x, y) = 0.$$

Note that the average price put: $G(x, y) = (K - y)^+$ and the average strike call: $G(x, y) = (x - y)^+$ satisfy this condition. Then we have

$$\lim_{y \to \infty} v(t, x, y) \quad = \quad E(e^{-r(T-t)} \lim_{y \to \infty} G(S(T), \frac{y + \int_t^T S_u du}{T}) | S_t = x) = 0.$$

Thus we can set the condition

$$\lim_{y \to \infty} v(t, x, y) = 0. \tag{3}$$

Then we have the following PDE for the Asian option, *assuming the condition* $\lim_{y \to \infty} G(x, y) = 0$

$$\begin{aligned}
&-rv(t, x, y) + v_t(t, x, y) + v_x(t, x, y)rx + v_y(t, x, y)x + \frac{1}{2} v_{xx}(t, x, y)\sigma^2(t, x)x^2 = 0, \\
&0 < x, y < \infty, 0 \leq t < T; \\
&v(T, x, y) = G(x, \frac{y}{T}); \\
&v(t, 0, y) = e^{-r(T-t)} G(0, \frac{y}{T}); \\
&\lim_{y \to \infty} v(t, x, y) = 0.
\end{aligned}$$

But then what about the average price call: $G(x, y) = (y - K)^+$ and the average strike put: $G(x, y) = (y - x)^+$? Intuitively we want to take $\lim_{y \to -\infty} G(x, y) = 0$. However, with our current definition of Y_t, this does not make sense, since $Y_t \geq 0$. So we need to extend our model by defining:

$$Y_t = Y(0) + \int_0^t S(u) du,$$

where $Y(0)$ *is a valued specified by the option contract, which can be negative or positive or zero.*
The payoff function G becomes

$$G(S(T), S_{ave}(T)) = G(S(T), \frac{1}{T}[Y(0) + \int_0^T S_u du]).$$

and the option value at time t is

$$v(t, x, y) = E\Big\{e^{-r(T-t)} G(S(T), Y(T)) | S_t = x, Y_t = y\Big\},$$

as before. Adding a constant $Y(0)$ at time $t = 0$ clearly does not change the Markov property of V_t. Note that since $Y(0)$ can take any value (positive, negative, zero), y here also can take any value (positive, negative, zero).
In words, what we did here is just allow flexibility for dicussing our function $v(t, x, y)$ as $y \to -\infty$. But then arguing exactly as before, under the assumption that $\lim_{y \to -\infty} G(x, y) = 0$ we have

$$\lim_{y \to -\infty} v(t, x, y) \quad = \quad E\Big\{e^{-r(T-t)} \lim_{-y \to \infty} G(S(T), \frac{y + \int_t^T S_u du}{T}) | S_t = x\Big\} = 0.$$

Then we have the following PDE for the Asian option, *assuming the condition* $\lim_{y \to -\infty} G(x, y) = 0$

$$\begin{aligned}
&-rv(t, x, y) + v_t(t, x, y) + v_x(t, x, y)rx + v_y(t, x, y)x + \frac{1}{2} v_{xx}(t, x, y)\sigma^2(t, x)x^2 = 0, \\
&0 < x < \infty, -\infty < y < \infty, 0 \le t < T; \\
&v(T, x, y) = G(x, \frac{y}{T}); \\
&v(t, 0, y) = e^{-r(T-t)} G(0, \frac{y}{T}); \\
&\lim_{y \to \infty} v(t, x, y) = 0.
\end{aligned}$$

Note the change of domain for y *on the first equation. Now* y *is defined on* $(-\infty, \infty)$*, not just* $[0, \infty)$.