Pricing the Knock out Barrier option and Look back option via Expectation

Math 622

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1 Overview

In this lecture, we'll see how we can evaluate the expression

$$V_t = E^Q(e^{-r(T-t)}V_T \Big| \mathcal{F}_t),$$

where

$$V_T = (S_T - K)^+ \mathbf{1}_{\{\max_{[0,T]} S_t \leq B\}}$$

for knock-out barrier option, or

$$V_T = \max_{[0,T]} S_t - S_T,$$

for look-back option.

It is clear that to compute V_t in these expressions, we need to know the distribution of $\max_{[0,T]} S_t$. But since

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t},$$

the distribution of $\max_{[0,T]} S_t$ is closely related to the distribution of M_t, the running max of the Brownian motion:

$$M_t := \max_{u\in[0,t]} W_u.$$

Instead of computing the distribution of M_t by it self, we will see that it is easier to compute the joint distribution of M_t, W_t. The key for us to derive this joint distribution is via the reflection principle, which says a reflected Brownian motion is also a Brownian motion. Using this principle and a probability identity, we will derive the joint distribution of M_t, W_t. From that, we can derive the distribution of M_t as a marginal distribution. Finally, we'll see how we can apply this knowledge to evaluate V_t in the two expressions above.

2 The reflection principle

2.1 Definition

Let W_t be a Brownian motion w.r.t a filtration $\mathcal{F}(t)$ and τ a $\mathcal{F}(t)$ stopping time. We define

$$\begin{aligned} B^\tau &:= W_t, t \le \tau \\ &:= W(\tau) - [W_t - W(\tau)], t > \tau. \end{aligned}$$

That is B^τ is the same as W_t up to the random time τ and after time τ is obtained by reflecting W_t around the horizontal line $y = W(\tau)$. We say B^τ is a reflected Brownian motion at τ.

2.2 The reflection principle

Theorem 2.1. *The B^τ defined above is a $\mathcal{F}(t)$ Brownian motion.*

In words, the reflection principel says a refleted Brownian motion is a Brownian motion.
The heuristics of why the Theorem is true is
(i) The strong Markov property: $W_t - W(\tau)$ is a Brownian motion independent of $\mathcal{F}(\tau)$
and
(ii) The negative of a Brownian motion is also a Brownian motion. Thus before t, B^τ is a Brownian motion, after τ it is also a Brownian motion (although starting at $W(\tau)$ instead of at 0). The key is how to show when we go across τ the Brownian motion property is still preserved and we achieve that by Levy's characterization of Brownian motion.

Proof. Define

$$\begin{aligned} a(t) &= 1, t \le \tau \\ &= -1, t > \tau. \end{aligned}$$

That is

$$\begin{aligned} a(t) &= \mathbf{1}_{t\le\tau} - \mathbf{1}_{t>\tau} \\ &= \mathbf{1}_{t\le\tau} - (1 - \mathbf{1}_{t\le\tau}) \\ &= 2\mathbf{1}_{t\le\tau} - 1. \end{aligned}$$

It is easy then to see $a(t) \in \mathcal{F}(t), \forall t$ since τ is a stopping time. It is also bounded, hence is in L^2. Thus we can consider $\int_0^t a(s)dW_s$. We have

$$\begin{aligned} \int_0^t a(s)dW_s &= \int_0^t 2\mathbf{1}_{s\le\tau} dW_s - W_t \\ &= \int_0^t 2\mathbf{1}_{[0,\tau)}(s)dW_s - W_t \\ &= 2W(t\wedge\tau) - W_t = B^\tau(t). \end{aligned}$$

(Just consider what happens when $\tau \le t$ and $\tau > t$.)
Thus $B^\tau(t)$ is a martingale. Moreover, its quadratic variation is:

$$\langle B^\tau \rangle_t = \int_0^t \alpha^2(s)ds = t,$$

since $\alpha(s)$ is either 1 or -1. Thus by Levy's characterization, B^τ is a Brownian motion.

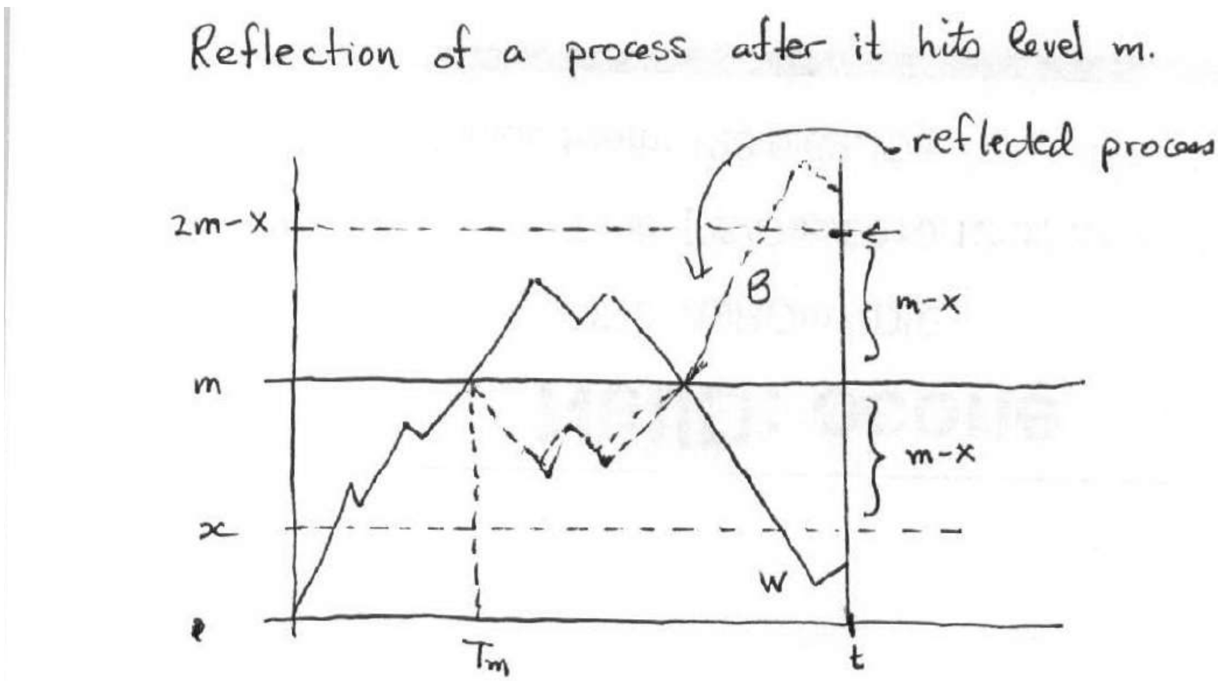
2.3 An important identity

Let W_t be a Brownian motion and $M_t := \max_{[0,t]} W_s$ its running maximum. The reflection principle helps us obtain the joint density between W_t and M_t through the following important identity: for $w \le m, m \ge 0$

$$\Big\{M_t > m, W_t < w\Big\} = \Big\{B_t > 2m - w\Big\},$$

where $B_t := B^{\tau_m}(t)$ is the Brownian motion obtained by reflecting W_t at time τ_m, the first hitting time of W_t to level m:

$$\tau_m := \inf\{t \ge 0 : W_t = m\}.$$



Remark 2.2. *Our goal with the identity is to use it to derive the joint density $f_t(m,w)$ of M_t, W_t, therefore we are only interested in considering $m \geq w$ and $m \geq 0$ because we always have $M_t \geq W_t$ and $M_t \geq W(0) = 0$.*

Proof. Proof of the identity
(i) Suppose $M_t > m$ and $W_t < w$. Then $M_t > m$ implies $\tau_m < t$ and hence

$$\begin{aligned} B_t &= 2W(\tau_m) - W_t \\ &= 2m - W_t > 2m - w. \end{aligned}$$

(ii) Suppose $B_t > 2m - w$. Then $B_t > m$ because $w \leq m$. So it cannot be the case that $B_t = W_t$ since that would imply $W_t > m$ and thus $\tau_m < t$, a contradiction to $B_t = W_t$ only when $t < \tau_m$. Thus $B_t = 2m - W_t$ and $\tau_m < t$ which implies $M_t > m$. Moreover,

$$B_t = 2m - W_t > 2m - w$$

implies $W_t < w$ and we are done.

2.4 Joint distribution of W_t and M_t

From the identiy above and the reflection principle (which implies B_t is a Brownian motion) we have

$$P(M_t > m, W_t < w) = P(B_t > 2m - w) = \int_{2m-w}^{\infty} \frac{e^{\frac{-x^2}{2t}}}{\sqrt{2\pi t}} dx.$$

If $f_t(m, w)$ is the joint density of (M_t, W_t) then

$$P(M_t > m, W_t < w) = \int_{-\infty}^{w} \int_{m}^{\infty} f_t(z, x) dz dx = \int_{2m-w}^{\infty} \frac{e^{\frac{-x^2}{2t}}}{\sqrt{2\pi t}} dx.$$

Thus by the Fundamental Theorem of Calculus, we get

$$\begin{aligned} f_t(m, w) &= -\frac{\partial^2}{\partial m \partial w} P(M_t > m, W_t < w) \\ &= \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} \mathbf{1}_{m \geq 0, w \leq m}. \end{aligned}$$

This has the following useful consequence: let $Z_t = 2M_t - W_t$. Then the joint density of (M_t, Z_t) is

$$g_t(m, z) = \frac{2z}{t\sqrt{2\pi t}} e^{-z^2/2t} \mathbf{1}_{\{m>0, z>m\}} = -2\frac{d}{dz} \frac{e^{-z^2/2t}}{\sqrt{2\pi t}} \mathbf{1}_{\{m>0, z>m\}}. \quad (1)$$

3 A useful function in evaluation of Barrier and Lookback options

3.1 Introduction

When computing the price of Knockout Barrier and Lookback Options, you'll see that because of the structure of the stock price, we'll usually end up computing an expression of the form

$$E\Big[\mathbf{1}_{\{W_s \geq k\}} \mathbf{1}_{\{M_s > b\}} e^{\alpha W_s + \beta M_s}\Big],$$

where α, β, k, b are general parameters that we can plug in depending on the option we're dealing with. Since this expression appears often in this context, we'll denote it by $H_s(\alpha, \beta, k, b)$, as a function of the unspecified parameters at a time s. That is

$$H_s(\alpha, \beta, k, b) := E\Big[\mathbf{1}_{\{W_s \geq k\}} \mathbf{1}_{\{M_s > b\}} e^{\alpha W_s + \beta M_s}\Big].$$

In the following sub-sections, we'll see how we can compute explicitly $H_s(\alpha, \beta, k, b)$ in some special case.

3.2 $H_s(\alpha, 0, k, b)$ when $0 \leq b \leq k$

Since $M_s \geq W_s$ we have if $W_s \geq k$ then $M_s \geq W_s \geq k \geq b$.
Thus

$$\Big\{W_s \geq k\Big\} \cap \Big\{M_s \geq b\Big\} = \Big\{W_s \geq k\Big\}.$$

In other words,

$$\mathbf{1}_{\{W_s \geq k\}} \mathbf{1}_{\{M_s > b\}} = \mathbf{1}_{\{W_s \geq k\}}.$$

So

$$H_s(\alpha, 0, k, b) = E\Big[\mathbf{1}_{\{W_s \geq k\}} e^{\alpha W_s}\Big] = e^{s\frac{\alpha^2}{2}} N\big(\frac{s\alpha - k}{\sqrt{s}}\big). \tag{2}$$

3.3 $H_s(\alpha, 0, k, b)$ when $k < b$

Theorem 3.1. *If $k < b$,*

$$H_s(\alpha, 0, k, b) = e^{s\frac{\alpha^2}{2}} \Big\{ N\big(\frac{s\alpha - b}{\sqrt{s}}\big) + e^{2\alpha b} \Big[N\big(\frac{-s\alpha - b}{\sqrt{s}}\big) - N\big(\frac{-s\alpha - 2b + k}{\sqrt{s}}\big) \Big] \Big\}.$$

Proof. Since $k < b$,

$$\begin{aligned} E\Big[\mathbf{1}_{\{W_s \geq k\}} \mathbf{1}_{\{M_s > b\}} e^{\alpha W_s}\Big] &= E\Big[\mathbf{1}_{\{W_s \geq b\}} \mathbf{1}_{\{M_s > b\}} e^{\alpha W_s}\Big] \\ &\quad + E\Big[\mathbf{1}_{\{k \leq W_s < b\}} \mathbf{1}_{\{M_s > b\}} e^{\alpha W_s}\Big]. \end{aligned}$$

Now

$$E\Big[\mathbf{1}_{\{W_s \geq b\}} \mathbf{1}_{\{M_s > b\}} e^{\alpha W_s}\Big] = H_s(\alpha, 0, b, b),$$

and we have found the expression for $H_s(\alpha, 0, b, b)$ in Section 2.1. As for the 2nd term, observe that

$$\Big\{k < W_s < b, M_s > b\Big\} = \Big\{W_s < b, M_s > b\Big\} \cap \Big\{k < W_s, M_s > b\Big\}.$$

We have showed that

$$\Big\{W_s < b, M_s > b\Big\} = \Big\{B^{\tau_b}(s) > b\Big\},$$

where B^{τ_b} is again W_t reflected at τ_b, the first hitting time of W_t to level b.

We claim that

$$\Big\{k < W_s, M_s > b\Big\} = \Big\{M_s > b, B^{\tau_b}(s) < 2b - k\Big\}.$$

(This is left as part of the homework).
Thus noting that $B^{\tau_b}(s) > b$ implies $M_s > b$ we get

$$\begin{aligned}\Big\{k < W_s < b, M_s > b\Big\} &= \Big\{B^{\tau_b}(s) > b\Big\} \cap \Big\{M_s > b, B^{\tau_b}(s) < 2b - k\Big\} \\ &= \Big\{b < B^{\tau_b}(s) < 2b - k\Big\}.\end{aligned}$$

We leave it as the other part of the homework to use this and (2) to complete the proof.

3.4 $H_s(\alpha, \beta, -\infty, b)$

Theorem 3.2.

$$\begin{aligned}H_s(\alpha, \beta, -\infty, b) &= \frac{\beta+\alpha}{\beta+2\alpha} 2e^{\frac{(\alpha+\beta)^2}{2}s} N\big(\frac{(\alpha+\beta)s - b}{\sqrt{s}}\big) \\ &\quad + \frac{2\alpha}{\beta+2\alpha} e^{\frac{\alpha^2}{2}s} e^{b(\beta+2\alpha)} N\big(-\frac{\alpha s + b}{\sqrt{s}}\big).\end{aligned}$$

Proof.
Let $Z_s = 2M_s - W_s$, so that $W_s = 2M_s - Z_s$. We will rewrite the expectation in the definition of $H_s(\alpha, \beta, -\infty, b)$ in terms of $Z(s)$ and M_s and use the joint density for these two random variables, which we stated above in (1). Thus,

$$\begin{aligned}H_s(\alpha, \beta, -\infty, b) = E\Bigg[\mathbf{1}_{\{M_s \geq b\}} e^{\alpha W(s) + \beta M_s}\Bigg] &= E\Bigg[\mathbf{1}_{\{M_s \geq b\}} e^{-\alpha Z(s) + (\beta+2\alpha) M_s}\Bigg] \\ &= \int_b^\infty e^{(\beta+2\alpha)m} \int_m^\infty e^{-\alpha z}\Bigg[-2\frac{d}{dz}\frac{e^{-z^2/2s}}{\sqrt{2\pi s}}\Bigg]\,dz\,dm. \qquad (3)\end{aligned}$$

By integration by parts, and then application of the formula

$$E\left[e^{aX}\mathbf{1}_{\{X \geq c\}}\right] = \int_c^\infty e^{ax} e^{-x^2/(2s)} \frac{dx}{\sqrt{2\pi s}} = e^{(a^2/2)s} N\left(\frac{as - c}{\sqrt{s}}\right), \qquad (4)$$

where X has Normal$(0, s)$ distribution, the inner integral is

$$2e^{-\alpha m}\frac{e^{-m^2/2s}}{\sqrt{2\pi s}} - 2\alpha \int_m^\infty e^{-\alpha z}\frac{e^{-z^2/2s}}{\sqrt{2\pi s}}\,dz = 2e^{-\alpha m}\frac{e^{-m^2/2s}}{\sqrt{2\pi s}} - 2\alpha e^{\frac{\alpha^2}{2}s} N\left(\frac{-\alpha s - m}{\sqrt{s}}\right)$$

Thus,

$$E\left[\mathbf{1}_{\{M_s\ge b\}}e^{\alpha W(s)+\beta M_s}\right] = 2\int_b^\infty e^{(\beta+\alpha)m}\frac{e^{-m^2/2s}}{\sqrt{2\pi s}}\,dm \\ -2\alpha e^{\frac{\alpha^2}{2}s}\int_b^\infty e^{(\beta+2\alpha)m}N\left(\frac{-\alpha s-m}{\sqrt{s}}\right)dm \qquad (5)$$

By applying (4) again, the first term is $2e^{\frac{(\beta+2\alpha)^2}{2}s}N\left(\frac{(\beta+2\alpha)s-b}{\sqrt{s}}\right)$. By integrating by parts and applying (4) yet again, the second term is

$$\frac{2\alpha}{\beta+2\alpha}\left[e^{((\alpha+\beta)^2/2)s}N\left(\frac{(\alpha+\beta)s-b}{\sqrt{s}}\right)-e^{(\alpha^2/2)s}e^{b(2\alpha+\beta)}N\left(\frac{-\alpha s-b}{\sqrt{s}}\right)\right]$$

By substituting these results in (5) one obtains the result.

4 Pricing Knock-out Barrier option via expectation

Let S_t satisfies

$$dS_t = rS_tdt + \sigma S_t dW_t.$$

Consider the Knock-out Barrier option with barrier b and strike price K:

$$V_T = (S_T-K)^+\mathbf{1}_{\{\max_{[0,T]}S_t\le B\}}.$$

The risk-neutral price V_t can be expressed as

$$\begin{aligned} V_t &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}V_T\Big|\mathcal{F}(t)\right] \\ &= \mathbf{1}_{\{\max_{u\in[0,t]}S_u\le B\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T-K)^+\mathbf{1}_{\{\max_{u\in[t,T]}S_u\le B\}}\Big|S_t\right]. \end{aligned}$$

To obtain an explicit formula for V_t, we need to evaluate

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T-K)^+\mathbf{1}_{\{\max_{u\in[t,T]}S_u\le B\}}\Big|S_t\right] \\ = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T-K)^+\Big|S_t\right]-\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T-K)^+\mathbf{1}_{\{\max_{u\in[t,T]}S_u> B\}}\Big|S_t\right]. \qquad (6)$$

Since $\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T-K)^+\Big|S_t\right]$ is already given by Black-Scholes formula, we only need to evaluate

$$\begin{aligned} w(t,x) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T-K)^+\mathbf{1}_{\{\max_{u\in[t,T]}S_u> B\}}\Big|S_t=x\right] \\ &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T-K)\mathbf{1}_{\{S_T\ge K\}}\mathbf{1}_{\{\max_{u\in[t,T]}S_u> B\}}\Big|S_t=x\right] \qquad (7) \end{aligned}$$

Remark 4.1. *The split in Equation* (6) *is to allow us to write* $w(t,x)$ *in the form of* $H_s(\alpha,\beta,k,b)$ *as we will see later.*

4.1 Step 1: A first rewrite of $w(t,x)$

Denote

$$\alpha := \frac{r - \frac{1}{2}\sigma^2}{\sigma}.$$

Then for $s \geq t$

$$S(s) = S_t \exp\Big[\sigma\big\{W_s - W_t + \alpha(s-t)\big\}\Big].$$

The term inside the exponential (modulo the σ) is just a Brownian motion with drift starting at time t. So we denote it by a new name to reflect this fact:

$$\begin{aligned}
\widehat{W}(u) &:= W(t+u) - W_t + \alpha u, u \geq 0 \\
\widehat{M}(u) &:= \max_{s\in[0,u]} \widehat{W}_s.
\end{aligned}$$

Note that for $s \geq t$

$$\max_{u\in[t,s]} S_u = S_t e^{\sigma \widehat{M}(s-t)}.$$

Then for $s \geq t$ we have

$$\begin{aligned}
S(s) &= S_t e^{\sigma \widehat{W}_{s-t}} \\
\mathbf{1}_{S(s)\geq K} &= \mathbf{1}_{\widehat{W}_{s-t} \geq \frac{\log(K/S_t)}{\sigma}} \\
\mathbf{1}_{\max_{u\in[t,s]} S_u > B} &= \mathbf{1}_{\widehat{M}(s-t) > \frac{\log(B/S_t)}{\sigma}}
\end{aligned}$$

Then substituting this into Equation (7), replacing $S_t = x$ gives

$$\begin{aligned}
w(t,x) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\big(xe^{\sigma\widehat{W}_{T-t}} - K\big)\mathbf{1}_{\{\widehat{W}_{T-t}\geq \frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}_{T-t} > \frac{\log(B/x)}{\sigma}\}}\Big] \\
&= xe^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[e^{\sigma\widehat{W}_{T-t}}\mathbf{1}_{\{\widehat{W}_{T-t}\geq \frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}_{T-t} > \frac{\log(B/x)}{\sigma}\}}\Big] \\
&\quad -Ke^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\mathbf{1}_{\{\widehat{W}_{T-t}\geq \frac{\log(K/x)}{\sigma}\}}\mathbf{1}_{\{\widehat{M}_{T-t} > \frac{\log(B/x)}{\sigma}\}}\Big].
\end{aligned} \tag{8}$$

Remark 4.2. *Note that the expression in* (8) *involves the distribution of a Brownian motion with drift and its running maximum. Studying Brownian motion with drift is inconvenient. But by applying a change of measure (via Girsanov's theorem), we can find a different measure such that under it,* $\widehat{W}$ *is a Brownian motion. So that's our next step.*

4.2 Apply Girsanov's Theorem to transform $\widehat{W}$ into a Brownian motion

Observe that there exists a Brownian motion $\widetilde{W}(u), 0 \leq u \leq T-t$, namely $\widetilde{W}_u = W_{t+u} - W_t$, such that

$$\widehat{W}(u) = \widetilde{W}(u) + \alpha u, u \in [0, T-t].$$

Since $\widehat{W}$ has drift term αt, our change of measure kernel is

$$Z_{T-t} = \exp[-\alpha \widetilde{W}(T-t) - \frac{\alpha^2}{2}(T-t)].$$

Denoting our original measure as $\mathbb{Q}$ and define

$$d\widehat{\mathbb{P}} := Z_{T-t} d\mathbb{Q},$$

then note that

$$\begin{aligned} d\mathbb{Q} &= Z_{T-t}^{-1} d\widehat{\mathbb{P}} \\ &= \exp[\alpha \widehat{W}_{T-t} - \frac{\alpha^2}{2}(T-t)], \end{aligned}$$

So that

$$\begin{aligned} w(t,x) &= x e^{-r(T-t)} \widehat{\mathbb{E}}\Big[e^{\sigma \widehat{W}_{T-t}} \mathbf{1}_{\{\widehat{W}_{T-t} \geq \frac{\log(K/x)}{\sigma}\}} \mathbf{1}_{\{\widehat{M}_{T-t} > \frac{\log(B/x)}{\sigma}\}} e^{\alpha \widehat{W}_{T-t} - \frac{\alpha^2}{2}(T-t)}\Big] \\ &\quad - K e^{-r(T-t)} \widehat{\mathbb{E}}\Big[\mathbf{1}_{\{\widehat{W}_{T-t} \geq \frac{\log(K/x)}{\sigma}\}} \mathbf{1}_{\{\widehat{M}_{T-t} > \frac{\log(B/x)}{\sigma}\}} e^{\alpha \widehat{W}_{T-t} - \frac{\alpha^2}{2}(T-t)}\Big] \\ &= x e^{-(r+\frac{\alpha^2}{2})(T-t)} \widehat{\mathbb{E}}\Big[e^{(\alpha+\sigma)\widehat{W}_{T-t}} \mathbf{1}_{\{\widehat{W}_{T-t} \geq \frac{\log(K/x)}{\sigma}\}} \mathbf{1}_{\{\widehat{M}_{T-t} > \frac{\log(B/x)}{\sigma}\}}\Big] \\ &\quad - K e^{-(r+\frac{\alpha^2}{2})(T-t)} \widehat{\mathbb{E}}\Big[e^{\alpha \widehat{W}_{T-t}} \mathbf{1}_{\{\widehat{W}_{T-t} \geq \frac{\log(K/x)}{\sigma}\}} \mathbf{1}_{\{\widehat{M}_{T-t} > \frac{\log(B/x)}{\sigma}\}}\Big], \end{aligned} \tag{9}$$

where now *what we have gained is $\widehat{W}$ is a Brownian motion under $\widehat{\mathbb{P}}$.*

4.3 Writing $w(t,x)$ in terms of $H_s(\alpha, \beta, k, b)$

Let W_t be a Brownian motion and $M_t := \max_{[0,t]} W_s$ its running maximum. Recall that we defined

$$H_s(\alpha, \beta, k, b) := E\Big[\mathbf{1}_{\{W_s \geq k\}} \mathbf{1}_{\{M_s > b\}} e^{\alpha W_s + \beta M_s}\Big].$$

Then we have

$$
\begin{aligned}
w(t,x) &= e^{-(r+\frac{\alpha^2}{2})(T-t)}\Big[xH_{T-t}\Big(\alpha+\sigma, 0, \frac{\log(K/x)}{\sigma}, \frac{\log(B/x)}{\sigma}\Big) \\
&\quad -KH_{T-t}\Big(\alpha, 0, \frac{\log(K/x)}{\sigma}, \frac{\log(B/x)}{\sigma}\Big)\Big].
\end{aligned}
$$

and the original Knockout Barrier option price is:

$$V_t = \mathbf{1}_{\{\max_{[0,t]} S_t \le B\}}\big[c(t,S_t) - w(t,S_t)\big],$$

where

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

is given by Black-Scholes formula.

5 Pricing Lookback Option via expectation

5.1 Preliminary discussion

Let S_t satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Consider the Knock-out Barrier option with barrier b and strike price K:

$$V_T = \max_{[0,T]} S_t - S_T.$$

The risk-neutral price V_t can be expressed as

$$
\begin{aligned}
V_t &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)}V_T\Big|\mathcal{F}(t)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)}\max_{[0,T]}\{S_t\} - S_T\Big|\mathcal{F}(t)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)}\max_{[0,T]}\{S_t\}\Big|\mathcal{F}(t)\Big] - S_t.
\end{aligned}
$$

Now to do further analysis (to reduce Conditional Expectation to an Expectation via the Independence Lemma) we may want to separate the term $\max_{[0,T]}\{S_t\}$ into some expression involving $S_u, u \in [0,t]$ and $S_u, u \in [t,T]$. One way to do this is

$$\max_{t\in[0,T]}\{S_t\} = \max_{u\in[0,t]}\{S_u\} \vee \max_{u\in[t,T]}\{S_u\}.$$

So far so good, but we need to do more work here, since the operator $\vee$ does not "factor out" of the conditional expectation (we cannot factor $\max_{[0,t]}\{S_t\}$ out of $E(.\Big|\mathcal{F}(t))$). Looking at this in another way, the running max of S_t:

$$Y_t = \max_{u\in[0,t]} S_u$$

is not a Markov process.

However, there is a usual approach in studying Markov process like this: If $X(t)$ is not a Markov process, by increasing the components of $X(t)$, we may still yet obtain a Markov process.

In this case, we consider the two-component process (S_t, Y_t) instead of just Y_t. Then for $s > t$

$$Y(s) = \max\{Y_t, \max_{[t,s]} S_u\} = \max\{Y_t, S_t e^{\sigma \widehat{M}(s-t)}\},$$

where recall that we defined in Section 1

$$\begin{aligned}
\alpha &:= \frac{r - \frac{1}{2}\sigma^2}{\sigma} \\
\widehat{W}(u) &:= W(t+u) - W_t + \alpha u, u \geq 0 \\
\widehat{M}(u) &:= \max_{s\in[0,u]} \widehat{W}_s.
\end{aligned}$$

Then since $\widehat{M}$ and $\widehat{W}$ are independent of $\mathcal{F}(t)$ under the risk neutral measure, we get that (S_t, Y_t) is a Markov process under this measure as well (how to reach this conclusion is left as a homework exercise).

We then have

$$\begin{aligned}
V_t &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)} V_T \Big| \mathcal{F}(t)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)} \max_{[0,T]}\{S_t\} \Big| \mathcal{F}(t)\Big] - S_t \\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)} \max\{Y_t, S_t e^{\sigma \widehat{M}_{T-t}}\} \Big| \mathcal{F}(t)\Big] - S_t \\
&= v(t, S_t, Y_t),
\end{aligned}$$

where

$$v(t,x,y) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\Big[\max(y, x e^{\sigma \widehat{M}_{T-t}})\Big] - x.$$

5.2 Apply Girsanov

Similar to the discussion in section 1, $v(t,x,y)$ involves the distribution of the running max of a Brownian motion with drift, so we want to apply Girsanov's theorem to transform it to a Brownian motion. The result is

$$v(t,x,y) = e^{-r(T-t)}\widehat{\mathbb{E}}\Big[e^{\alpha \widehat{W}_{T-t} - \frac{\alpha^2}{2}(T-t)} \max(y, xe^{\sigma \widehat{M}_{T-t}})\Big] - x,$$

where $\widehat{W}$ now is a Brownian motion under $\widehat{\mathbb{P}}$.
Note that the expression inside expectation is not (yet) of the form provided by the function $H_s(\alpha,\beta,k,b)$. Noting the fact that $x>0$ since it is the stock price S_t, we have

$$\begin{aligned} \max(y, xe^{\sigma \widehat{M}_{T-t}}) = y \quad &\text{if} \quad \widehat{M}_{T-t} < \frac{1}{\sigma}\log(y/x) \\ \max(y, xe^{\sigma \widehat{M}_{T-t}}) = xe^{\sigma \widehat{M}_{T-t}} \quad &\text{if} \quad \widehat{M}_{T-t} \geq \frac{1}{\sigma}\log(y/x). \end{aligned}$$

Denoting

$$b := \frac{1}{\sigma}\log(y/x),$$

and note that the domain of interest for $v(t,x,y)$ is $y \geq x > 0$ thus $b \geq 0$. Then

$$\begin{aligned} \max(y, xe^{\sigma \widehat{M}_{T-t}}) &= y\mathbf{1}_{\widehat{M}_{T-t}<b} + xe^{\sigma \widehat{M}_{T-t}}\mathbf{1}_{\widehat{M}_{T-t}\geq b} \\ &= y + \Big[xe^{\sigma \widehat{M}_{T-t}} - y\Big]\mathbf{1}_{\widehat{M}_{T-t}\geq b}. \end{aligned}$$

Plug this back into the expectation, coupled with the fact that

$$\widehat{\mathbb{E}}\Big[ye^{\alpha \widehat{W}_{T-t} - \frac{\alpha^2}{2}(T-t)}\Big] = y,$$

after simplification we have

$$\begin{aligned} v(t,x,y) &= e^{-r(T-t)}y - x + xe^{-(r+\frac{\alpha^2}{2})(T-t)}\widehat{\mathbb{E}}\Big[\mathbf{1}_{\{\widehat{M}_{T-t}\geq b\}}e^{\alpha \widehat{W}_{T-t} + \sigma \widehat{M}_{T-t}}\Big] \\ &\quad -ye^{-(r+\frac{\alpha^2}{2})(T-t)}\widehat{\mathbb{E}}\Big[\mathbf{1}_{\{\widehat{M}_{T-t}\geq b\}}e^{\alpha \widehat{W}_{T-t}}\Big] \\ &= e^{-r(T-t)}y - x + xe^{-(r+\frac{\alpha^2}{2})(T-t)}H_{T-t}\big(\alpha, \sigma, -\infty, b\big) \\ &\quad -ye^{-(r+\frac{\alpha^2}{2})(T-t)}H_{T-t}\big(\alpha, 0, -\infty, b\big). \end{aligned} \tag{10}$$

Remark 5.1. *You may question why we go into such length to derive the closed form expression for the value of the Barrier or Lookback Option via expectation. An alternative, as you may have already known, is ti simulate the paths of* S_t *and take the average over the simulated paths to obtain an approximation for the expectation. However, the work that we have done, for example, in expressing* $v(t, x, y)$ *in the form of* (10) *can be very helpful in increasing the efficiency of the computation. We have "simplified" the computation (not in the expression, of course, but in the actual computation time). The reason is the function* $H_s(\alpha, \beta, k, b)$ *is found explicitly via the cumulative distribution of the standard normal, which we have very efficient algorithms to compute. On the other hand, as you can already imagine, the efficiency of simulating the paths of* S_t*, also taking into account its running max or when it reaches the barrier and then take the average might not be as good as just computing the probability of a Normal distribution.*