PDE for knock out barrier option

Math 622 - Spring 2015

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1 Introduction

In Chapter 7, we consider the risk neutral price for various exotic options:
(i) Knock out Barrier option:

$$V_T = (S_T - K)^+ \mathbf{1}_{\{\max_{[0,T]} S_t \leq b\}}.$$

(ii) Lookback option:

$$V_T = \max_{[0,T]} S_t - S(T).$$

(iii) Asian option:

$$V_T = \left(\frac{1}{T} \int_0^T S_t dt - K \right)^+$$

The risk-neutral price $V(t)$ in all of these cases can be expressed as

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} V_T | \mathcal{F}(t) \right].$$

To analyze V_t further, it is tempting to write $V(t) = v(t, S(t))$ for some function $v(t, x)$ and start deriving what equation $v(t, x)$ has to satisfy. However, this is incorrect.
Recall that the basis for us to say there exists such a function $v(t, x)$ is because of the Indepndence lemma, which in turns rely on the fact that we can write

$$S_T = S_t \times (\text{ something independent of } \mathcal{F}(t))$$

and we were working with European option, which only depends on S_T.

That is not the case here: all these three exotic options are *path dependent*, i.e. the expression for V_T involves the values of $S_t, 0 \leq t \leq T$, not just S_T. So apriori, it is not clear that we can find such a $v(t, x)$. Indeed, for the Lookback and Asian option, we will see that the correct function to deal with is $v(t, x, y)$, not $v(t, x)$, where we need to add another component $Y(t)$ to $S(t)$ so that the joint process $S(t), Y(t)$ have the necessary Markov property.

For the Knockout Barrier option, *the key idea is to analyze the behavior of S_t upto the first time it hits the barrier.* This time, as you may have known, is a *stopping time* with respect to the filtration generated by S_t. So we will begin by reviewing stopping time and its properties. We will then show how we can derive the PDE for the price V_t of a Knockout Barrier option using stopping time.

2 Stopping times

2.1 Motivation

In financial math, very often and quite naturally, we study random decisions, such as when to exercise your right to buy an option (American call option), or when to accept an offer for the house you are selling (imagine you're putting your house on a market and offer comes in for how much the buyer is willing to pay for the house, which is random). These decisions involve a random time (the time you decide to take action). The time is random because obviously it depends on the path of the stock's price, or of the offers, which are random.

However, there is a common important feature in both cases here: *your decision of when to take action cannot depend on future information.* Mathematically, if we denote $\mathcal{F}(t)$ as the stream of information available to you at time t, and the random time when you take action is τ, then we require:

$$\{\tau \leq t\} \in \mathcal{F}(t).$$

The event $\{\tau \leq t\}$ means you have taken action on or before time t. The event being $\in \mathcal{F}(t)$ then means your decision of taking action on or before time t entirely depends on the information up to time t, i.e. it does not involve future information. Such τ is called a stopping time and it is an important concept to study.

2.2 Some preliminary

2.2.1 Discrete vs continuous time

We can model time in 2 ways. Discrete: consider time $n = 0, 1, 2, ..., N$ where N is our terminal time. Continuous: consider time $t \in [0, T]$, where T is our terminal time. Stopping times are defined in both contexts. Generally speaking, discrete time is "easier" to analyze (don't take this statement too literally). The models we will study in Chapter 7,8 are in continuous time. Generally, most of the statements about stopping times have similar versions in both discrete and continuous times. But when one works in continuous time, it is good to pay attention because there will be subtleties that are not present in discrete time.

2.2.2 Filtration, sigma-algebra and the flow of information

We denote $\mathcal{F}(t), t \in [0, T]$ to be the filtration in the time interval $[0, T]$, which represents the information we have available up to time t. We require:
(i) Each $\mathcal{F}(t)$ is a sigma-algebra.
(ii) If $s < t$ then $\mathcal{F}(s) \subseteq \mathcal{F}(t)$.
Condition (i) is about the closure property of $\mathcal{F}(t)$: if $A_i, i = 1, 2, ...$ is a countable sequence of events (meaning the number of events can potentially be infinite) in $\mathcal{F}(t)$, then A_i^c (not A_i), $\cup_{i=1}^{\infty} A_i$ (some of A_i has happened), $\cap_{i=1}^{\infty} A_i$ (all of A_i have happened) are also in $\mathcal{F}(t)$. We also require $\emptyset, \Omega \in \mathcal{F}(t)$.
Condition (ii) is about the flow of information, intuitively at the present time t we must also have knowledge of the information of the past up to time s as well.
Sometimes we have $\mathcal{F}(0) = \{\emptyset, \Omega\}$. This means any event at time 0 is deterministic. In terms of a random process, this means the process starts out at a deterministic point x, instead of having a random initial distribution.
We can also consider $\mathcal{F}(n), n = 0, 1, ..., N$ as the discrete analog of continuous time filtration. The requirements are the same.

2.2.3 Stopping time definition

Definition 2.1. *Let τ be a random variable taking values in $[0, T]$ (resp. $\{0, 1, ..., N\}$). We say τ is a stopping time with respect to $\mathcal{F}(t)$ (resp. $\mathcal{F}(n)$) if for* *all* $t \in [0, T]$ *(resp. for all* $n = 0, 1, ..., N$*)*

$$\begin{aligned} \{\tau \leq t\} &\in \mathcal{F}(t) \\ (\text{ resp. } \{\tau \leq n\} &\in \mathcal{F}(n)). \end{aligned}$$

Remark 2.2. *Note that the notion of a stopping time is tied to a filtration (similar to the notion of a martingale). It could happen that* τ *is a stopping time with respect to a filtration* $\mathcal{F}(t)$ *but not a stopping time with respect to another, smaller filtration* $\mathcal{G}(t) \subseteq \mathcal{F}(t)$.

2.2.4 First important difference between discrete and continuous time

Consider the discrete time. Since if τ is a $\mathcal{F}(n)$ stopping time then $\{\tau < n\} = \{\tau \leq n - 1\} \in \mathcal{F}(n-1) \subseteq \mathcal{F}(n)$, we have

$$\{\tau \geq n\} = \{\tau < n\}^c \in \mathcal{F}(n)$$

Hence

$$\{\tau = n\} = \{\tau \leq n\} \cap \{\tau \geq n\} \in \mathcal{F}(n).$$

Conversely if $\{\tau = n\} \in \mathcal{F}(n)$ *for all* n then $\{\tau \leq n\} = \cup_{i=0}^{n} \{\tau = i\} \in \mathcal{F}(n)$, for all n as well. So we can use either conditions: $\{\tau = n\} \in \mathcal{F}(n)$ or $\{\tau \leq n\} \in \mathcal{F}(n)$ as definition for stopping time in discrete time.
Now consider the continuous time. By the property of stopping time listed below, it is also true that $\{\tau < t\} \in \mathcal{F}(t)$. So $\{\tau \geq t\} = \{\tau < t\}^c \in \mathcal{F}(t)$. Therefore, if τ is a stopping time then

$$\{\tau = t\} = \{\tau \leq t\} \cap \{\tau \geq t\} \in \mathcal{F}(t).$$

However, *it is NOT true that if* $\{\tau = t\} \in \mathcal{F}(t)$ *for all* t *then* $\{\tau \leq t\} \in \mathcal{F}(t)$. The reason is because in continuous time, we need to write

$$\{\tau \leq t\} = \cup_{0 \leq s \leq t} \{\tau = s\},$$

and the RHS involves an *uncountable* union of events, which doesn't have to be contained in the sigma algebra. This explains the choice of using $\{\tau \leq t\} \in \mathcal{F}(t)$ as the definition for continuous time.

2.2.5 Some properties of stopping time

Lemma 2.3. *Let τ_1, τ_2 be stopping times with respect to $\mathcal{F}(t)$. Then*
(i) $\{\tau_1 < t\} \in \mathcal{F}(t), \forall 0 \le t \le T$;
(ii) $\min(\tau_1, \tau_2)$ and $\max(\tau_1, \tau_2)$ are stopping times with respect to $\mathcal{F}(t)$.

Property (i) follows from the fact that

$$\{\tau_1 < t\} = \cup_{n=1}^{\infty} \{\tau_1 \le t - \frac{1}{n}\},$$

and $\{\tau_1 \le t - \frac{1}{n}\} \in \mathcal{F}(t - \frac{1}{n}) \subseteq \mathcal{F}(t), \forall n$. Property ii is left as homework exercise.

2.3 Some important examples

Example 2.4. *Jump time of a Poisson process*
Let $N(t)$ be a Poisson process. Then

$$\tau_k := \inf\{t \ge 0 : N(t) = k\}$$

are stopping times with respect to $\mathcal{F}^N(t)$.

Reason: $\{\tau_k \le t\}$ means the k^{th} jump happened at or before t. But that is the same as at time t, $N(t) \ge k$. Thus

$$\{\tau_k \le t\} = \{N(t) \ge k\} \in \mathcal{F}(t).$$

Example 2.5. *First hitting time to a point of Brownian motion*
Let $b > 0$ be fixed. Define

$$T_b := \inf\{t \ge 0 : W(t) = b\}$$

to be the first time $W(t)$ hits the level b. Note also the convention that $\inf \emptyset = \infty$, that is if $W(t)$ never hits b then we set $T_b = \infty$. Then T_b is a stopping time with respect to $\mathcal{F}^W(t)$.

The reasoning here is more complicated. Note that $\{T_b \le t\}$ means $W(.)$ has hit b at or before time t. But we cannot infer any property of $W(t)$ (say $W(t) \ge b$ based on this information) because W is *not monotone.*
It is better to look at the complement: $\{T_b > t\}$ which means $W(.)$ has NOT hit b at or before t, which since $W(.)$ starts at 0 at time 0 is equivalent to

$W(s) < b, 0 < s < t$, the information of which intuitively belongs to $\mathcal{F}(t)$. But this is not rigorous, since again there are uncountably many points s in $[0, t]$.
To fix this, we note that a continuous function is uniquely determined by its values on the rationals, which is countable. Combine these facts we can write

$$\begin{aligned}\{T_b > t\} &= \{W(s) < b, 0 \leq s \leq t\} = \cup_{i=1}^{n}\{W(s) \leq b - \frac{1}{n}, 0 \leq s \leq t\} \\ &= \cup_{i=1}^{n}\{W(s) \leq b - \frac{1}{n}, s \in [0, t] \cap \mathbb{Q}\} \\ &= \cup_{i=1}^{n} \cap_{s \in \mathbb{Q}} \{W(s) \leq b - \frac{1}{n}\},\end{aligned}$$

and it follows that $\{T_b > t\} \in \mathcal{F}(t)$. Note the subtle fact here that we need to transition from $W(s) < b$ to $W(s) \leq b - \frac{1}{n}$ for some n. The reason is this: if $W(s) < b$ for all s rationals, we can only conclude that $W(s) \leq b$ for all s. But $W(s) \leq b$ for all s rational if and only if $W(s) \leq b$ for all s.
We did not use any special property of Brownian motion besides the fact that it has continuous paths. So

Example 2.6. *First hitting time to a point of a continuous process*
Let $b > 0$ be fixed. Let $X(t)$ be a process starting at 0 with continuous paths. Define

$$T_b := \inf\{t \geq 0 : X(t) = b\}$$

to be the first time $X(t)$ hits the level b. Then T_b is a stopping time with respect to $\mathcal{F}^X(t)$.

Example 2.7. *Non example: last hitting time*
Let $b > 0$ be fixed. Let $X(t)$ be a process starting at 0 with continuous paths. Define

$$T_b := \sup\{t \geq 0 : X(t) = b\}$$

to be the last time $X(t)$ hits the level b. Then T_b may NOT be a stopping time with respect to $\mathcal{F}^X(t)$.

The reason is this: $\{T_b \leq t\}$ means the last time $X(t)$ hits b is at or before time t. But it is impossible to know whether $X(t)$ will hit b again unless we observe the future paths of $X(t)$, which is forbidden for a stopping time definition. There is an exception: if we know that $X(t)$ is monotone, then once it hits b it will not hit b again. But this is probably the only exception.

Example 2.8. *First hitting time to an open set of a continuous process*
Let $b > 0$ be fixed. Let $X(t)$ be a process starting at 0 with continuous paths. Define

$$S_b := \inf\{t \geq 0 : X(t) > b\}$$

to be the first time $X(t)$ hits the open set (b, ∞). Then S_b may NOT be a stopping time with respect to $\mathcal{F}^X(t)$.

The reason is very subtle here. It is tempting to write

$$\begin{aligned} \{S_b > t\} &= \{X_s \leq b, 0 \leq s \leq t\} = \{X_s \leq b, s \in \mathbb{Q}\} \\ &= \cap_{s \in \mathbb{Q}} \{X_s \leq b\}, \end{aligned}$$

therefore $\{S_b > t\} \in \mathcal{F}(t)$ and S_b is a stopping time. What happens is the first equality is incorrect, and it is because of the definition of infimum. It could be the case that at time t, $X(t) = b$ and immediately after t, X crosses over b. Then in this case $S_b = t$ and the event we describe is still in the RHS of the above equation. In other words,

$$\{S_b \geq t\} = \{X_s \leq b, 0 \leq s \leq t\}$$

and we don't have the right inequality to work with here. But note the fact that S is almost a stopping time. We call it an optional time here.

Remark 2.9. *Another useful way to think of the above situation is to imagine 2 possible paths of $X(s)$: one path ω hits b at time t and crosses over. The other ω'* **follows the exact same path up to time** t*, hits b at time t and immediately reflects down, and let's say never comes back to level b. Then $S_b(\omega) = t$ and $S_b(\omega') = \infty$. Since the two paths are* **the same up to time** t*, it is impossible to tell the event $S_b = t$ by observing $\mathcal{F}(t)$. So S_b cannot be a stopping time. This can be used as a useful, even though non-rigorous criterion to determine whether a random time is a stopping time.*

2.4 Strong Markov property of Brownian motion

It is a well-known fact of Brownian motion that it has independent and stationary increment: if $t > s$ then $W(t) - W(s)$ is independent of $\mathcal{F}(s)$ and has distribution $N(0, t - s)$. In particular, this implies that $W(t) - W(s)$ is a Brownian motion independent of $\mathcal{F}(s)$.

What is interesting is if we replace s by a stopping time, all of these results still hold, except for the technical issue of defining what $\mathcal{F}(\tau)$ means. For our purpose, it is enough to think of $\mathcal{F}(\tau)$ as the sigma algebra containing all information before time τ and we have the following:

Theorem 2.10. *Strong Markov property*
Let W be a Brownian motion and $\mathcal{F}(t)$ a filtration for W. Let τ be a $\mathcal{F}(t)$ stopping time. Then $W(\tau+u)-W(\tau), u \geq 0$ is a Brownian motion independent of all the information in the filtration $\mathcal{F}(t)$ before time τ.

This theorem is called the Strong Markov property because it implies that the Markov property of Brownian motion can be applied to a stopping time as well. Indeed, if we accept, in addition to the strong Markov property, the fact that $W(\tau) \in \mathcal{F}(\tau)$ then by the Independence Lemma:

$$E[f\big(W(\tau+u)\big)|\mathcal{F}(\tau)] = g(W(\tau)),$$

where

$$g(x) = E[f\big(x+W_u\big)].$$

2.5 An important result in the case of Brownian motion

Let W_t be a Brownian motion starting at 0. Let $b > 0$ and define

$$\begin{aligned} T_b &:= \inf\{t \geq 0 : W(t) = b\} \\ S_b &:= \inf\{t \geq 0 : W(t) > b\}. \end{aligned}$$

Then it is clear that $T_b \leq S_b$. We have also remarked above that T_b is a stopping time with respect to the Brownian filtration while S_b is only an optional time. A very interesting result here is that even though these times are different in nature, the probability of the event that they differ is 0. That is

Lemma 2.11. *Let $W(t)$ be a Brownian motion, then*

$$\mathbb{P}(T_b = S_b) = 1.$$

Remark 2.12. *The above Lemma says that S_b is equal to T_b up to sets of measure 0. Therefore, if we include sets of measure 0 in $\mathcal{F}(t)$, for all t, a procedure called augmentation of filtration, then S_b is a stopping time with respect to the augmented filtration.*

Proof. We present the idea of the proof of this result here. For complete details, see e.g. [1] problem 7.19.
The proof of the Lemma (2.11) depends on two other important results about the path of Brownian and the Brownian filtration. They are as followed:
a. Blumenthal 0-1 law : Let $\mathcal{F}_t$ be a filtration generated by a Brownian motion. Let E be an event in the sigma algebra $\mathcal{F}_{0+}$. Then $P(E) = 0$ or $P(E) = 1$.
Remark: $E \in \mathcal{F}_{0+}$ means that we can have the information of E by observing infinitesimally into the future beyond the time 0, but not necessarily exactly at time 0.
b. Infinite crossing property: Let W_t be a Brownian motion starting at 0. Then for **any** $\varepsilon > 0$,

$$P(W_t \text{ crosses 0 infinitely often in the time interval } [0, \varepsilon]) = 1.$$

We will take the 0-1 law as a fact. The infinite crossing property can be explained using the 0-1 law as followed. Define

$$\begin{aligned} T_0^+ &:= \inf\{t \geq 0 : W_t > 0\} \\ T_0^- &:= \inf\{t \geq 0 : W_t < 0\}. \end{aligned}$$

Then arguing as we did before, we can show the events $\{T_0^+ = 0\}$ and $\{T_0^- = 0\}$ are in $\mathcal{F}_{0+}$. Then by Blumenthal 0-1 law, $P(T_0^+ = 0) = 0$ or $P(T_0^+ = 0) = 1$, similarly for T_0^-. By symmetry of the distribution of Brownian motion ($-W_t$ is a Brownian motion iff W_t is a Brownian motion) we also have

$$P(T_0^+ = 0) = P(T_0^- = 0).$$

Therefore, they must both be 0 or both be 1. Now suppose that both $P(T_0^+ = 0) = P(T_0^- = 0) = 0$. That must mean with positive probability we can find an $\varepsilon > 0$ so that $W_t = 0$ identically on $[0, \varepsilon]$. But this is impossible since this implies that with positive probability, the quadratic variation of W_t on $[0, \varepsilon]$ is equal to 0. Thus we must conclude

$$P(T_0^+ = 0) = P(T_0^- = 0) = 1.$$

That is with probability 1, W_t crosses 0 infinitely often in the time interval $[0, \varepsilon]$.
We will now show $\mathbb{P}(T_b = S_b) = 1$ using these two facts and the strong Markov property of W_t. Since T_b is a stopping time, $\widetilde{W_t^b} := W_t - W_{T_b}$ is a Brownian motion for $t \geq T_b$.

Suppose that in contrary to the conclusion of the Lemma, $P(S_b > T_b) > 0$. Then with positive probability, there is a time interval (namely on $[T_b, S_b]$) so that $\widetilde{W}_t^b \leq 0$ on $[T_b, S_b]$. That is $\widetilde{W}^b$ does not cross 0 on the time interval $[T_b, S_b]$. But this contradicts fact b) we mentioned above. This establishes the Lemma.

2.6 Stopped processes

Definition 2.13. *Given a stochastic X and a random time τ, we define the stopped process X at time τ as*

$$\begin{aligned} X(t \wedge \tau(\omega))(\omega) &:= X(t)(\omega), t \leq \tau(\omega) \\ &:= X_{\tau(\omega)}(\omega), t \geq T(\omega). \end{aligned}$$

When τ is a stopping time and X is a martingale then the stopped process is also a martingale via the following theorem:

Theorem 2.14. *Let $M(t)$ be a martingale with respect to $\mathcal{F}(t)$ with càdlàg paths. Let τ be a stopping time with respect to $\mathcal{F}(t)$. Then $M(t \wedge \tau)$ is also a martingale with respect to $\mathcal{F}(t)$.*

This theorem has a discrete time analog:

Theorem 2.15. *Let $M(n)$ be a martingale with respect to $\mathcal{F}(n)$ and τ a $\mathcal{F}(n)$ stopping time. Then $X(t \wedge n)$ is also a martingale with respect to $\mathcal{F}(n)$.*

In particular, in the continuous time, when M is a stochastic integral against Brownian motion, then the stopped processed $M(t \wedge \tau)$ is also a martingale when τ is a stopping time. But in this case, we also have an interesting representation of the stopped stochastic integral via the following theorem.

Theorem 2.16. *Let $\mathcal{F}(t)$ be a filtration and $W(t)$ a $\mathcal{F}(t)$ Brownian motion. Let α be an adapted process to $\mathcal{F}(t)$ such that $\int_0^t \alpha(s)dW(s)$ is well-defined. Let τ be a $\mathcal{F}(t)$ stopping time. Denote $M(t) := \int_0^t \alpha(s)dW(s)$. Then $M(t \wedge \tau)$ is a $\mathcal{F}(t)$ martingale. Moreover,*

$$M(t \wedge \tau) = \int_0^{t \wedge \tau} \alpha(s)dW(s) = \int_0^t \mathbf{1}_{[0,\tau)}(s)dW(s).$$

The following corollary is an immediate consequence of the above theorem:

Corollary 2.17. *Let S_t have the dynamics:*

$$dS_t = \alpha_t dt + \sigma_t dW_t.$$

Then for a stopping time τ

$$\begin{aligned} S_{t\wedge\tau} &= S_0 + \int_0^{t\wedge\tau} \alpha(s)ds + \int_0^{t\wedge\tau} \sigma_s dW(s) \\ &= S_0 + \int_0^t \mathbf{1}_{[0,\tau)}(s)\alpha_s ds + \int_0^t \mathbf{1}_{[0,\tau)}(s)\sigma_s dW(s). \end{aligned}$$

2.7 Generalization of Lemma (2.11) to Ito processes

Lemma (2.11) can be generalized to general Ito process: process that can be written as a Rieman integral plus an Ito integral. The intuition here is that the Ito integral has path property similar to that of Brownian motion: very irregular. On the other hand, the Rieman integral has a differentiable ("regular") path. So when the process $X(t)$ hits b, the effect of the stochastic integral part would win out and cause the process to enter b as in the presence of only a Brownian motion.

Theorem 2.18. *Let*

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \sigma(s)dW(s),$$

and suppose that $\mathbb{P}(\sigma(t) \neq 0) = 1$ for all t.
Define

$$\begin{aligned} T_b &:= \inf\{t : X(t) = b\} \\ S_b &:= \inf\{t : X(t) > b\}. \end{aligned}$$

Then $\mathbb{P}(T_b = S_b) = 1$.

3 Knock-out Barrier option

3.1 The goal

Let $S(t)$ satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Consider the Knock-out Barrier option with barrier b and strike price K:

$$V_T = (S_T - K)^+ \mathbf{1}_{\{\max_{[0,T]} S_t \le b\}}.$$

Note: Necessarily we require $K < b$ and $S(0) < b$ so that $\mathbb{P}(V_T > 0) > 0$.
The risk neutral price $V(t)$ can be written as:

$$V(t) = E\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[0,T]} S_t \le b\}} | \mathcal{F}(t)\Big].$$

Our goal is to find a function $v(t,x)$ so that $V_t = v(t, S_t)$ and then apply Ito's formula to $e^{-rt}v(t, S_t)$ to find a PDE that $v(t,x)$ satisfies. It is not immediately clear that this can be achieved, as the expression of V_t above involves the term $\mathbf{1}_{\{\max_{[0,T]} S_t \le b\}}$. Indeed as we shall see there is no such function $v(t,x)$ so that the equality

$$V_t = v(t, S_t)$$

holds true *for all* $0 \le t \le T$.
However, we observe that *before* S_t hits b, it is believable that the option value V_t should be just a function of (t, S_t) (think about the factors that you would use to value V_t in a real life situation before the stock hits the barrier). *After* S_t hits b, the option value V_t stays constant, namely it takes value 0. That is, the option value V_t should be a function of (t, S_t) *upto* the random time T_b, the first time S_t hits b. In other words, we are looking to find a function $v(t,x)$ so that

$$V_t = v(t \wedge T_b, S_{t\wedge T_b}), t \in [0,T].$$

This is what we will establish rigorously in several steps in the following section. This equality will also help us establish a PDE for $v(t,x)$ since by the result of the section (2.6), we can apply Ito's formula to $v(t \wedge T_b, S_{t\wedge T_b})$. Note, however, that here we are investigating the dynamics of $v(t, S_t)$ on the time interval $[0, T_b]$. Thus our PDE will not have the usual domain as the one in classical Black-Scholes PDE.

3.2 The steps

We proceed to establish

$$V_t = v(t \wedge T_b, S_{t\wedge T_b}),$$

for some function $v(t,x)$ through several steps.
(i) Write $\mathbf{1}_{\{\max_{[0,T]} S_t \le b\}}$ in terms of $S_u, 0 \le u \le t$ and $S_u, t \le u \le T$.

The reason is we want to apply the Independence Lemma (or quote the Markov property of $S(t)$), so heuristically we want to "separate the past and the future". We already know how to do this with S_T. So we apply the same principle to the new term $\mathbf{1}_{\{\max_{[0,T]} S_t \le b\}}$.
This is accomplished as followed:

$$\mathbf{1}_{\{\max_{[0,T]} S_t \le b\}} = \mathbf{1}_{\{\max_{[0,t]} S_u \le b\}} \mathbf{1}_{\{\max_{[t,T]} S_u \le b\}}.$$

It is easy to see why the equality is true: the maximum of the whole path does not exceed b if and only if its maximum on each time interval does not exceed b.
(ii) Recognizing that $\mathbf{1}_{\{\max_{[0,t]} S_u \le b\}} \in \mathcal{F}(t)$, so it can be factored out of $E(.|\mathcal{F}(t))$.
(iii) Define

$$\begin{aligned} \tau_b &:= \inf\{t \ge 0 : S(t) > b\} \wedge T \\ T_b &:= \inf\{t \ge 0 : S(t) = b\} \wedge T \end{aligned}$$

Recall that $P(T_b = \tau_b) = 1$. And so with probability 1:

$$\{\max_{[0,t]} S_u \le b\} = \{\tau_b \ge t\} = \{T_b \ge t\}.$$

The change from τ_b to T_b might seem unimportant and non-intuitive. But it is to apply the optinal stopping theorem for martingale, see the section on the derivation of the PDE below.
(iv) Combine (ii) and (iii) we get

$$V(t) = \mathbf{1}_{T_b \ge t} E\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[t,T]} S_u \le b\}} | \mathcal{F}(t)\Big].$$

(v) Since

$$S(T) = S(t) e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W(T) - W(t))}$$

and

$$\max_{[t,T]} S_u = S_t \max_{[t,T]} e^{(r - \frac{1}{2}\sigma^2)(u-t) + \sigma(W(u) - W(t))},$$

note that $\max_{[t,T]} e^{(r - \frac{1}{2}\sigma^2)(u-t) + \sigma(W(u) - W(t))}$ is independent of $\mathcal{F}(t)$, by the Independence Lemma, we get

$$E\Big[e^{-r(T-t)}(S_T - K)^+ \mathbf{1}_{\{\max_{[t,T]} S_u \le b\}} | \mathcal{F}(t)\Big] = v(t, S(t)).$$

where

$$v(t,x) := E\Big[e^{-r(T-t)}\big(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))} - K\big)^+ \times \mathbf{1}_{\{x\max_{[t,T]} e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W(u)-W(t))}\leq b\}}\Big].$$

(vi) (*Crucial point*)

$$V(t) = \mathbf{1}_{T_b\geq t}v(t,S(t)) = v(t,S(t\wedge T_b)).$$

Indeed if $T_b \geq t$ then LHS $= v(t,S(t))$ and $t\wedge T_b = t$ so the RHS $= v(t,S(t))$ and the equality is true.
If $T_b < t$ then LHS $= 0$. $t\wedge T_b = T_b$ so that $S(t\wedge T_b) = b$. Moreover, with probability 1:

$$b\max_{[t,T]} e^{r(u-t)+\sigma(W(u)-W(t))} > b$$

Indeed, if we denote $X(u) := r(u-t)+\sigma(W(u)-W(t)), u\in[t,T]$ then $X(t) = 0$ and by property of Brownian motion,

$$P(X(u)\leq 0, \forall u\in[t,T]) = 0.$$

So there must exist $u\in(t,T]$, $X(u)>0$ and at that point u, $be^{X(u)} > b$. Thus

$$v(t,S(t\wedge T_b)) = v(t,b) = \Big[e^{-r(T-t)}\big(be^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))} - K\big)^+ \mathbf{1}_{\{b\max_{[t,T]} e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W(u)-W(t))}\leq b\}}\Big] = 0,$$

and so the RHS $= 0$ as well.

(vii) From the above, we see that the function $v(t,x)$ satisfies $v(t,b) = 0$ for all t. Therefore, it follows that

$$v(\tau,b) = 0,$$

for all stopping time τ taking values in $[0,T]$. From which we derive that

$$v(t\wedge T_b, S_{t\wedge T_b}) = v(t, S_{t\wedge T_b}).$$

Indeed, for $t < T_b$ the equalities are clear. For $t > T_b$, then
$v(T_b, S_{T_b}) = v(T_b, b) = 0 = v(t,b)$ so the equalities are also true in this case. Therefore,

$$V_t = v(t,S(t\wedge T_b)) = v(t\wedge T_b, S(t\wedge T_b)).$$

3.3 Derivation of the PDE

3.3.1 Derivation

We have

$$\begin{aligned} S(t\wedge T_b) &= S(0)+\int_0^{t\wedge T_b} rS(u)du+\int_0^{t\wedge T_b}\sigma S(u)dW(u) \\ &= S(0)+\int_0^t \mathbf{1}_{[0,T_b)} rS(u)du+\int_0^t \mathbf{1}_{[0,T_b)}\sigma S(u)dW(u). \end{aligned}$$

Apply Ito's formula to $e^{-rt}v(t,S_{t\wedge T_b})$ (where we look at $v(t,S_{t\wedge T_b})$ as a deterministic function of t and the stopped process $S_{t\wedge T_b}$), we have

$$\begin{aligned} e^{-rt}V_t &= e^{-rt}v(t,S_{t\wedge T_b}) = v(0,S_0)+\int_0^t e^{-ru}\big[-rv+v_t+\mathbf{1}_{[0,T_b)}(u)rS_u v_x \\ &\quad +\frac{1}{2}\mathbf{1}_{[0,T_b)}(u)\sigma^2 S_u^2 v_{xx}\big]du+\int_0^t \mathbf{1}_{[0,T_b)}(u)e^{-ru}\sigma S(u)v_x dW_u, \end{aligned}$$

where for all functions v we understood as $v(t,S_t)$, similarly for v_t, v_x, v_{xx}.
Note that this is where the importance of using T_b instead of τ_b is. The stochastic integral

$$\int_0^t e^{-ru}\mathbf{1}_{[0,T_b)}\sigma S(u)v_x dW_u = \int_0^{t\wedge T_b} e^{-ru}\sigma S(u)v_x dW_u$$

is a martingale since T_b is a stopping time. If we use τ_b here we cannot make the same conclusion for τ_b is not a stopping time.
We do not want to set the dt term equal to 0 yet, because in the dt integral above, some terms include $\mathbf{1}_{[0,T_b)}$ (from the stopped process S_t) and some don't.
But this is easy to fix, since $e^{-rt}V_t$ being a martingale implies $e^{-r(t\wedge T_b)}V_{t\wedge T_b}$ is also a martingale. And it's easily seen that

$$\begin{aligned} e^{-r(t\wedge T_b)}V_{t\wedge T_b} &= v(0,S_0)+\int_0^t \mathbf{1}_{[0,T_b)}(u)e^{-ru}\big[-rv+v_t+rS_u v_x \\ &\quad +\frac{1}{2}\sigma^2 S_u^2 v_{xx}\big]du+\int_0^t \mathbf{1}_{[0,T_b)}e^{-ru}\sigma S(u)v_x dW_u, \end{aligned}$$

Therefore we conclude

$$\mathbf{1}_{[0,T_b)}(u)\big[-rv(u,S_u)+v_t(u,S_u)+rS_u v_x(u,S_u)+\frac{1}{2}\sigma^2 S_u^2 v_{xx}(u,S_u)\big]=0.$$

3.3.2 Domain of the PDE

The equality

$$\mathbf{1}_{[0,T_b)}(u)\big[-rv(u,S_u)+v_t(u,S_u)+rS_uv_x(u,S_u)+\frac{1}{2}\sigma^2S_u^2v_{xx}(u,S_u)\big]=0$$

does NOT permit us to conclude

$$-rv(t,x)+v_t(t,x)+rS_uv_x(t,x)+\frac{1}{2}\sigma^2S_u^2v_{xx}(t,x)\big]=0,$$

for all t,x.

The reason is we can only cancel out the term $\mathbf{1}_{[0,T_b)}(u)$ when it is NOT zero, which is the same as when $0<S_u\leq b$.

Thus the domain for our PDE is $[0,T]\times[0,b]$, which is *different* from the domain we used to work on for European call option: $[0,T]\times[0,\infty)$. One of the effect is that we will have *boundary conditions* for our PDE at $x=0$ and $x=b$.

Moreover, note that $v(t,0)=0$ since if $S(t)$ ever hits 0 it will stay there. $v(t,B)=0$ was explaind in step (vi). These are the boundary conditions for v. We also have the terminal condition $v(T,x)=(x-K)^+$ as usual.

Thus, the PDE that v must satisfy is:

$$\begin{aligned} v_t-rv+rxv_x+\frac{1}{2}x^2\sigma^2v_{xx} &= 0, 0\leq t<T, 0<x<b \\ v(t,0)=v(t,b) &= 0 \\ v(T,x) &= (x-K)^+. \end{aligned}$$

References

[1] Karatzas, Shreve. *Brownian motion and stochastic calculus.* Vol. 113. Springer Science and Business Media, 1991.