Pricing European call option in jump diffusion models

Math 622 - Spring 2015

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1 Pricing via risk neutral expectation

1.1 Model with Poisson noise

1.1.1 Change of measure

Let N_t be a Poisson process with rate λ. Supose we model the stock price as

$$dS_t = \alpha S_t dt + \sigma S_{t-} dM(t),$$

where $M(t) = N_t - \lambda t$ is a $\mathbb{P}$-martingale. Note that here the only random source of S_t is from the jump process N_t.

From section 9 of lecture note 1, we also have

$$S_t = S(0) \exp[(\alpha - \lambda\sigma)t + \log(1+\sigma)N_t].$$

Let $r > 0$ be the interest rate. We want to find $\mathbb{Q}$ such that $e^{-rt}S_t$ is a $\mathbb{Q}$ martingale. If that is the case, since

$$dS_t = rS_t dt + \sigma S_{t-}(dN_t - [\lambda - \frac{\alpha - r}{\sigma}]dt)$$

clearly we need N_t to be a Poisson process with rate $\widetilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$. Since $\widetilde{\lambda}$ must be positive, a necessary condition (which implies no arbitrage for the model of S_t) is

$$\lambda - \frac{\alpha - r}{\sigma} > 0.$$

We then define

$$\begin{aligned} d\mathbb{Q} &= Z(T)d\mathbb{P}; \\ Z(t) &= \exp[\log\left(\frac{\widetilde{\lambda}}{\lambda}\right) N_t - (\widetilde{\lambda} - \lambda)t)]. \end{aligned}$$

Note that under $\mathbb{Q}$, we write the dynamics of S_t as

$$dS_t = rS_t dt + \sigma S_{t-} d\widetilde{M}(t),$$

where $\widetilde{M}(t) = N_t - \widetilde{\lambda} t$ is a $\mathbb{Q}$-martingale, which is equivalent to

$$S_t = S(0)\exp[(r - \widetilde{\lambda}\sigma)t + \log(1+\sigma)N_t].$$

1.1.2 Pricing of European call

Let $V(t)$ denote the risk-neutral price of a European Call paying $V(T) = (S_T - K)^+$ at time T. Then by the risk neutral pricing formula, we have

$$V(t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}(t)\big].$$

It remains to find an expression for $V(t)$. Clearly

$$S_T = S_t \exp[(r - \widetilde{\lambda}\sigma)(T-t) + \log(1+\sigma)(N_T - N_t)].$$

So by the Independence Lemma, (Shreve's Lemma (2.3.4)), we only need to evaluate

$$c(t,x) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(xe^{(r-\widetilde{\lambda}\sigma)(T-t)+\log(1+\sigma)(N_T-N_t)} - K\right)^+\right],$$

then we have $V(t) = c(t, S_t)$.

Since $N_T - N_t$ has distribution Poisson($\widetilde{\lambda}(T-t)$) under $\mathbb{Q}$, $c(t,x)$ has the expression as an infinite sum, see Shreve's formula (11.7.3). We won't reproduce it here.

1.2 Model with compound Poisson noise

1.2.1 Change of measure

Suppose now that

$$dS_t = \alpha S_t dt + \sigma S_{t-} dM(t),$$

where $M(t) = Q(t) - mt$ is a compensated compound Poisson process under $\mathbb{P}$. Under the risk neutral probability $\mathbb{Q}$,

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_{t-} d\widetilde{M}(t) \\ &= (r - \sigma\widetilde{m})S_t dt + \sigma S_{t-} dQ_t. \end{aligned}$$

So clearly we need

(i) $Q(t)$ to be a compound Poisson process under $\mathbb{Q}$ with $\mathbb{E}^{\mathbb{Q}}(Q(1)) = \widetilde{m}$.

(ii) $r - \sigma\widetilde{m} = \alpha - \sigma m$.

Note that (ii) gives an equation for $\widetilde{m}$. If $Q(t) = \sum_{i=1}^{N_t} Y_i$ and under $\mathbb{Q}$, N_t is a Poisson process with rate $\widetilde{\lambda}$ and $\mathbb{E}(Y_i) = \widetilde{\mu}$ then

$$\widetilde{m} = \widetilde{\lambda}\widetilde{\mu}.$$

So (ii) also gives an equation for $\widetilde{\lambda}$ and $\widetilde{f}$, the distribution of Y_i under $\mathbb{Q}$. From the change of measure sections, we have seen how to choose $Z(T)$ such that the conditions (i) and (ii) are satisfied. Note that this choice may not be unique, as generally equation (ii) has more than 1 unknowns. However, there is also a restriction on the solution $\widetilde{\lambda} > 0$. So a simple application of linear algebra result to conclude that there are infinitely many choices of risk neutral measures is not correct.

1.2.2 Pricing of European call option

Observe that

$$dS_t = (r - \sigma\widetilde{m})S_t dt + \sigma S_{t-} dQ_t$$

has the solution

$$\begin{aligned} S_t &= S(0)e^{(r-\sigma\widetilde{m})t} \prod_{0<s\le t} (1 + \sigma\Delta Q(s)) \\ &= S(0)e^{(r-\sigma\widetilde{m})t} \prod_{i=1}^{N_t} (1 + \sigma Y_i). \end{aligned}$$

Also for $t < T$

$$S_T = S_t e^{(r-\sigma\widetilde{m})(T-t)} \prod_{i=N_t+1}^{N_T} (1 + \sigma Y_i).$$

Observe *the important fact* that $\prod_{i=N_t+1}^{N_T}(1 + \sigma Y_i)$ *is independent of* $\mathcal{F}(t)$, where $\mathcal{F}(t)$ is a filtration for $Q(t)$. We give an explanation in the next subsection.

Thus $V(t)$, the risk-neutral price of a European Call paying $V(T) = (S_T - K)^+$ at time T for this model is

$$\begin{aligned} V(t) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}(t)\big] \\ &= c(t, S_t), \end{aligned}$$

where

$$c(t,x) \;:=\; e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[[xe^{(r-\sigma\widetilde{m})(T-t)} \prod_{i=N_t+1}^{N_T} (1+\sigma Y_i) - K]^+\Big].$$

Since Y_i are independent of $N_T - N_t$, again we can condition on $N_T - N_t = j, j = 1, 2, ...$ to get

$$c(t,x) \;=\; e^{-r(T-t)} \sum_{j=0}^{\infty} \kappa(j,x) e^{-\widetilde{\lambda}(T-t)} \frac{[\widetilde{\lambda}(T-t)]^j}{j!},$$

where

$$\kappa(j,x) = \mathbb{E}^{\mathbb{Q}}\Big[\Big(xe^{(r-\sigma\widetilde{m})(T-t)} \prod_{i=1}^{j}(1+\sigma Y_i) - K\Big)^+\Big].$$

1.2.3 The independence of $\prod_{i=N_t+1}^{N_T}(1+\sigma Y_i)$ from $\mathcal{F}(t)$

In the derivation above, we claim that $\prod_{i=N_t+1}^{N_T}(1+\sigma Y_i)$ from $\mathcal{F}(t)$. This is not totally obvious since the terms involve N_t, which is in $\mathcal{F}(t)$. However, intuitively you can see that this is believable because there are $N_T - N_t$ terms in the product, which consists of Y_i's. Both $N_T - N_t$ and Y_i's are independent of $\mathcal{F}(t)$.

Rigorously, we use the result mentioned in Lecture 2. That is if $E(e^{uX}|\mathcal{F}) = E(e^{uX})$ for all $u \in \mathbb{R}$ then X is independent of $\mathcal{F}$. We verify that this is the case here. That is we want to show

$$E\Big(e^{u\prod_{i=N_t+1}^{N_T}(1+\sigma Y_i)}|\mathcal{F}_t\Big) = E\Big(e^{u\prod_{i=N_t+1}^{N_T}(1+\sigma Y_i)}\Big), \forall u \in \mathbb{R}.$$

Observe that the above expression would be complicated to handle. But we can simplify it by noting that we can instead show the independence of

$$\log\Big(\prod_{i=N_t+1}^{N_T}(1+\sigma Y_i)\Big) = \sum_{i=N_t+1}^{N_T} \log(1+\sigma Y_i)$$

with $\mathcal{F}_t$. Moreover,

$$e^{u\sum_{i=N_t+1}^{N_T}\log(1+\sigma Y_i)} = \prod_{i=N_t+1}^{N_T} e^{u\log(1+\sigma Y_i)}.$$

Thus we see that we can just prove this general claim for our purpose: let $X_1, X_2, \cdots$ be i.i.d and be independent of $N_t, t > 0$. Then

$$E\Big(\prod_{i=N_t+1}^{N_T} X_i|\mathcal{F}_t\Big) = E\Big(\prod_{i=N_t+1}^{N_T} X_i\Big) = E\Big(\prod_{i=1}^{N_T-N_t} X_i\Big).$$

This indeed will be the statement we'll prove for the rest of this proof. We have

$$E\Big(\prod_{i=N_t+1}^{N_T} X_i|\mathcal{F}(t)\Big) = E\Big(\prod_{i=1}^{N_T-N_t} X_{i+N_t}|\mathcal{F}(t)\Big).$$

Since $N_T - N_t$ is independent of $\mathcal{F}(t)$, by the Independence lemma,

$$\mathbb{E}\big[\prod_{i=1}^{N_T-N_t} X_{i+N_t}|\mathcal{F}(t)\big] = f(N_t),$$

where

$$f(k) = \mathbb{E}\big[\prod_{i=1}^{N_T-N_t} X_{i+k}\big].$$

We'll be done if we can show

$$f(k) = f(0) = \mathbb{E}\big[\prod_{i=1}^{N_T-N_t} X_i\big].$$

Note that

$$\begin{aligned}
\mathbb{E}\big[\prod_{i=1}^{N_T-N_t} X_{i+k}\big] &= \sum_j \mathbb{E}\big[\prod_{i=1}^{N_T-N_t} X_{i+k}|N_T - N_t = j\big]P(N_T - N_t = j) \\
&= \sum_j \mathbb{E}\big[\prod_{i=1}^{j} X_{i+k}|N_T - N_t = j\big]P(N_T - N_t = j) \\
&= \sum_j \mathbb{E}\big[\prod_{i=1}^{j} X_{i+k}\big]P(N_T - N_t = j) \\
&= \sum_j \Big\{\mathbb{E}\big[X_1\big]\Big\}^j P(N_T - N_t = j),
\end{aligned}$$

where the third equality is because of the independence of X_i's and $N_T - N_t$ and the fourth equality is because of the identical distribution of X_i's.

Using the same conditioning technique, we can also show

$$\mathbb{E}[\prod_{i=1}^{N_T-N_t} X_i] = \sum_j \Big\{\mathbb{E}[X_1]\Big\}^j P(N_T - N_t = j).$$

Thus $f(k) = f(0)$ as required.

1.3 Model with Brownian motion and compound Poisson noise

1.3.1 Change of measure

Suppose now that

$$dS_t = \alpha S_t dt + \sigma S_{t-} dW(t) + S_{t-} dM(t),$$

where $M(t) = Q(t) - mt$ is a compensated compound Poisson process under $\mathbb{P}$. Under the risk neutral probability $\mathbb{Q}$,

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_{t-} d\widetilde{W}(t) + S_{t-} d\widetilde{M}(t) \\ &= (r - \widetilde{m}) S_t dt + \sigma S_{t-} d\widetilde{W}(t) + S_{t-} dQ_t, \end{aligned}$$

where $\widetilde{W}(t) := W(t) + \theta t$ is a $\mathbb{Q}$ Brownian motion and $Q(t)$ is compound Poisson with $\mathbb{E}^{\mathbb{Q}}(Q(1)) = \widetilde{m}$.

Thus the equation that θ and $\widetilde{m}$ have to satisfy is

$$r + \sigma\theta - \widetilde{m} = \alpha - m.$$

Solving this equation for θ and $\widetilde{m}$ and use the change of measure result discussed above, we can find $\mathbb{Q}$ such that $e^{-rt} S_t$ is a $\mathbb{Q}$ - martingale.

1.3.2 Pricing of European call

Observe that

$$dS_t = (r - \widetilde{m}) S_t dt + \sigma S_{t-} d\widetilde{W}(t) + S_{t-} dQ_t,$$

has the solution

$$S_t = S(0)\exp\Big[(r - \widetilde{m} - \frac{1}{2}\sigma^2)t + \sigma\widetilde{W}(t)\Big]\prod_{i=1}^{N_t}(1+Y_i).$$

Hence for $t < T$,

$$S_T = S_t \exp\Big[(r - \widetilde{m} - \frac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{W}(T) - \widetilde{W}(t))\Big]\prod_{i=N_t+1}^{N_T}(1+Y_i),$$

where we have the *independence* of $\widetilde{W}(T) - \widetilde{W}(t)$ and $\prod_{i=N_t+1}^{N_T}(1+Y_i)$ with respect to $\mathcal{F}(t)$ and *also with respect to each other.*

Thus $V(t)$, the risk-neutral price of a European Call paying $V(T) = (S_T - K)^+$ at time T for this model is

$$\begin{aligned} V(t) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}(t)\big] \\ &= c(t, S_t), \end{aligned}$$

where

$$c(t,x) := e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\Big(xe^{(r-\widetilde{m}-\frac{1}{2}\sigma^2)(T-t)+\sigma(\widetilde{W}(T)-\widetilde{W}(t))}\prod_{i=N_t+1}^{N_T}(1+Y_i) - K\Big)^+\Big].$$

To find an expression for $c(t,x)$, we first condition on $\prod_{i=N_t+1}^{N_T}(1+Y_i)$ and use the independence lemma to define a function $\kappa(t,x)$ as

$$\kappa(t,x) := e^{-rt}\mathbb{E}^{\mathbb{Q}}\Big[\Big(xe^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Y} - K\Big)^+\Big],$$

where Y has standard normal distribution. Note that we have an explicit expression for $\kappa(t,x)$ from the Black-Scholes formula. Then

$$c(t,x) = \mathbb{E}^{\mathbb{Q}}\big[\kappa(T-t, xe^{-\widetilde{m}(T-t)}\prod_{i=N_t+1}^{N_T}(1+Y_i))\big].$$

Now again conditioning on $N_T - N_t = j$ and using the independence between $Y_i's$ and $N_T - N_t$ we have

$$c(t,x) = \sum_{j=0}^{\infty} e^{-\widetilde{\lambda}(T-t)}\frac{(\widetilde{\lambda}(T-t))^j}{j!}\mathbb{E}^{\mathbb{Q}}\big[\kappa(T-t, xe^{-\widetilde{m}(T-t)}\prod_{i=1}^{j}(1+Y_i))\big].$$

2 Pricing via partial differential difference equations

2.1 Heuristic

Suppose S_t satisfies

$$dS_t = \alpha S_t dt + \sigma S_t dM(t),$$

where $M(t) = N_t - \lambda t$ is a compensated Poisson process under $\mathbb{P}$.

From the change of measure section, we learned that under the risk neutral measure $\mathbb{Q}$, S_t has the dynamic:

$$dS_t = (r - \widetilde{\lambda}\sigma)S_t dt + \sigma S_{t-} dN_t,$$

where $\widetilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$ and N is a Poisson process with rate $\widetilde{\lambda}$ under $\mathbb{Q}$.

The call option price $V(t)$, where $V(T) = (S_T - K)^+$ can be written as

$$\begin{aligned} V(t) &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}(t)\Big] \\ &= c(t, S_t), \end{aligned}$$

where

$$c(t,x) := e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[(xe^{(r-\widetilde{\lambda}\sigma)(T-t)+\log(1+\sigma)(N_T - N_t)} - K)^+\Big].$$

As in the Black-Scholes model, we want to derive an equation that $c(t,x)$ satisfies. *The key principle* here is to apply Ito's formula to $e^{-rt}c(t,S_t)$ to achieve

$$de^{-rt}c(t,S_t) = f(t, c(t,S_t))dt + \text{ something } dM(t),$$

where $M(t)$ *is a* $\mathbb{Q}$*-martingale.* Then the equation that we look for is

$$f(t, c(t, S_t)) = 0.$$

The reason is that $e^{-rt}c(t,S_t)$ is a $\mathbb{Q}$-martingale by definition. Therefore, its drift has to be 0.

2.2 Model with Poisson noise

Suppose S_t satisfies

$$dS_t = \alpha S_t dt + \sigma S_t dM(t),$$

where $M(t) = N_t - \lambda t$ is a compensated Poisson process under $\mathbb{P}$.

Apply Ito's formula to $e^{-rt}c(t, S_t)$, recognizing there is no Brownian motion component, we have

$$\begin{aligned} e^{-rt}c(t, S_t) &= \int_0^t -re^{-ru}c(u, S_u)du + e^{-ru}\frac{\partial}{\partial t}c(u, S_u)du + e^{-ru}\frac{\partial}{\partial x}c(u, S_u)dS^c(u) \\ &\quad + \sum_{0<u\leq t} e^{-ru}[c(u, S_u) - c(u-, S_{u-})] \\ &= \int_0^t e^{-ru}\Big[-rc(u, S_u) + \frac{\partial}{\partial t}c(u, S_u) + \frac{\partial}{\partial x}c(t, S_u)(r - \widetilde{\lambda}\sigma)S_u\Big]du \\ &\quad + \sum_{0<u\leq t} e^{-ru}[c(u, S_u) - c(u, S_{u-})]. \end{aligned}$$

We need to rewrite $\sum_{0<u\leq t} e^{-ru}[c(u, S_u) - c(u, S_{u-})]$ as it is *not in differential form*. Two key observations will help us here:

(i) $S_u = (1 + \sigma\Delta N(u))S_{u-} = (1 + \sigma)S_{u-}$.

(ii) $c(u, S_u)$ jumps at the same points as S_u, which in turn jumps at the same points as $N(u)$. Again keep in mind that $\Delta N(u) = 1$.

Thus

$$\begin{aligned} \sum_{0<u\leq t} e^{-ru}[c(u, S_u) - c(u, S_{u-})] &= \sum_{0<u\leq t} e^{-ru}[c(u, S_{u-}(1 + \sigma)) - c(u, S_{u-})] \\ = \int_0^t e^{-ru}[c(u, S_{u-}(1 + \sigma)) - c(u, S_{u-})]dN(u), \end{aligned}$$

where the first equality uses observations (i) and second equality uses observation (ii).

Putting all these together gives

$$\begin{aligned} e^{-rt}c(t, S_t) &= \int_0^t e^{-ru}\Big[-rc(u, S_u) + \frac{\partial}{\partial t}c(u, S_u) + \frac{\partial}{\partial x}c(t, S_u)(r - \widetilde{\lambda}\sigma)S_u\Big]du \\ &\quad + \int_0^t e^{-ru}[c(u, S_{u-}(1 + \sigma)) - c(u, S_{u-})]dN(u). \end{aligned}$$

The last thing to do is to change $dN(u)$ to $dM(u)$ for some martingale M. This is easy: we only need to subtract and add $\widetilde{\lambda}du$ to $dN(u)$. So finally

$$\begin{aligned} e^{-rt}c(t,S_t) &= \int_0^t e^{-ru}\Big[-rc(u,S_u)+\frac{\partial}{\partial t}c(u,S_u)+\frac{\partial}{\partial x}c(t,S_u)(r-\widetilde{\lambda}\sigma)S_u \\ &\quad +[c(u,S_{u-}(1+\sigma))-c(u,S_{u-})]\widetilde{\lambda}\Big]du \\ &\quad +\int_0^t e^{-ru}[c(u,S_{u-}(1+\sigma))-c(u,S_{u-})]dM(u) \\ &= \int_0^t e^{-ru}\Big[-rc(u,S_u)+\frac{\partial}{\partial t}c(u,S_u)+\frac{\partial}{\partial x}c(t,S_u)(r-\widetilde{\lambda}\sigma)S_u \\ &\quad +[c(u,S_u(1+\sigma))-c(u,S_u)]\widetilde{\lambda}\Big]du \\ &\quad +\int_0^t e^{-ru}[c(u,S_{u-}(1+\sigma))-c(u-,S_{u-})]dM(u), \end{aligned}$$

where in the second equality we uses the fact that we are integrating with respect to du so using S_{u-} or S_u gives the same result.

Now apply the principle in Section (2.1) we get

Theorem 2.1. *The call option price $c(t,x)$ in the model of this section satisfies the differential difference equation*

$$\begin{aligned} -rc(t,x) &+ \frac{\partial}{\partial t}c(t,x)+(r-\widetilde{\lambda}\sigma)x\frac{\partial}{\partial x}c(t,x) \\ &+ \widetilde{\lambda}[c(t,x(1+\sigma))-c(t,x)]=0, 0\le t<T, x>0 \\ c(T,x) &= (x-K)^+, x>0. \end{aligned}$$

2.3 Model with compound Poisson noise

Suppose S_t has the dynamic:

$$dS_t=(r-\widetilde{m}\sigma)S_t dt+\sigma S_{t-}dQ_t,$$

where $Q(t)$ is a compound Poisson process with rate $\mathbb{E}^{\mathbb{Q}}(Q(1))=1$. We also assume that $Q(t)=\sum_{i=1}^{N_t}Y_i$ where each Y_i takes discrete distribution with values $y_1, y_2, ..., y_m$.

Following the same procedure as the above section, apply Ito's formula to $e^{-rt}c(t,S_t)$

gives

$$\begin{aligned} e^{-rt}c(t,S_t) &= \int_0^t -re^{-ru}c(u,S_u)du + e^{-ru}\frac{\partial}{\partial t}c(u,S_u)du + e^{-ru}\frac{\partial}{\partial x}c(u,S_u)dS^c(u) \\ &\quad + \sum_{0<u\le t} e^{-ru}[c(u,S_u) - c(u-,S_{u-})] \\ &= \int_0^t e^{-ru}\Big[-rc(u,S_u) + \frac{\partial}{\partial t}c(u,S_u) + \frac{\partial}{\partial x}c(t,S_u)(r-\widetilde{m}\sigma)S_u\Big]du \\ &\quad + \sum_{0<u\le t} e^{-ru}[c(u,S_u) - c(u,S_{u-})]. \end{aligned}$$

Now by the Poisson process decomposition, we can write

$$Q(t) = \sum_{i=1}^m y_i N_i(t),$$

where each $N_i(t)$ is a Poisson process with rate $\widetilde{\lambda}_i, i = 1,...,m$ under $\mathbb{Q}$. An important fact here is that since N_i's are independent, *they do not jump at the same time.* So at all jump point of Q:

$$1 + \sigma\Delta Q(t) = 1 + \sigma y_i \Delta N_i(t), \text{ for some } i.$$

Thus we have,

$$\begin{aligned} &\sum_{0<u\le t} e^{-ru}[c(u,S_u) - c(u,S_{u-})] = \sum_{0<u\le N_t} e^{-ru}[c(u,S_{u-}(1+\sigma\Delta Q_u)) - c(u,S_{u-})] \\ &= \sum_{i=1}^m \Big[\sum_{0<u\le t} e^{-ru}[c(u,S_{u-}(1+\sigma y_i)) - c(u,S_{u-})]\Delta N_i(u)\Big] \\ &= \sum_{i=1}^m \Big[\int_0^t e^{-ru}[c(u,S_{u-}(1+\sigma y_i)) - c(u,S_{u-})]dN_i(u)\Big]. \end{aligned}$$

So

$$\begin{aligned} e^{-rt}c(t,S_t) &= \int_0^t e^{-ru}\Big[-rc(u,S_u) + \frac{\partial}{\partial t}c(u,S_u) + \frac{\partial}{\partial x}c(t,S_u)(r-\widetilde{m}\sigma)S_u \\ &\quad + \sum_{i=1}^m [c(u,S_u(1+\sigma y_i)) - c(u,S_u)]\widetilde{\lambda}_i\Big]du \\ &\quad + \int_0^t e^{-ru}[c(u,S_u) - c(u,S_{u-})]dM(u), \end{aligned}$$

where

$$M(t) = \sum_{i=1}^{m} N_i(t) - \widetilde{\lambda}_i t$$

is a $\mathbb{Q}$-martingale.

Setting the dt part to be 0 gives the following:

Theorem 2.2. *The call option price $c(t,x)$ in the model of this section satisfies the differential difference equation*

$$\begin{aligned} -rc(t,x) &+ \frac{\partial}{\partial t}c(t,x) + (r - \widetilde{m}\sigma)x\frac{\partial}{\partial x}c(t,x) \\ &+ \sum_{i=1}^{m}[c(t,x(1+\sigma y_i)) - c(t,x)]\widetilde{\lambda}_i = 0, 0 \le t < T, x > 0 \\ c(T,x) &= (x-K)^+, x > 0. \end{aligned}$$

2.4 Model with Brownian motion and compound Poisson noise

Suppose S_t has the dynamic:

$$dSt = (r - \widetilde{m})S_t dt + S_{t-}dQ_t + \sigma S_t d\widetilde{W}(t),$$

where $Q(t)$ is a compound Poisson process with rate $\mathbb{E}^{\mathbb{Q}}(Q(1)) = 1$ and $\widetilde{W}(t)$ is a $\mathbb{Q}$ Brownian motion. We also assume that $Q(t) = \sum_{i=1}^{N_t} Y_i$ where each Y_i takes discrete distribution with values $y_1, y_2, ..., y_m$.

Following the same procedure as the above section, apply Ito's formula to $e^{-rt}c(t,S_t)$ gives

$$\begin{aligned} e^{-rt}c(t,S_t) &= \int_0^t -re^{-ru}c(u,S_u)du + e^{-ru}\frac{\partial}{\partial t}c(u,S_u)du + e^{-ru}\frac{\partial}{\partial x}c(u,S_u)dS^c(u) \\ &+\frac{1}{2}e^{-ru}\frac{\partial^2}{\partial x^2}c(u,S_u)\sigma^2 S^2(u)du + \sum_{0<u\le t} e^{-ru}[c(u,S_u) - c(u-,S_{u-})] \\ &= \int_0^t e^{-ru}\Big[- rc(u,S_u) + \frac{\partial}{\partial t}c(u,S_u) + \frac{\partial}{\partial x}c(t,S_u)(r-\widetilde{m})S_u \\ &+\frac{1}{2}\frac{\partial^2}{\partial x^2}c(u,S_u)\sigma^2 S^2(u)\Big]du \\ &+\int_0^t e^{-ru}\frac{\partial}{\partial x}c(t,S_u)S_u d\widetilde{W}(u) + \sum_{0<u\le t} e^{-ru}[c(u,S_u) - c(u,S_{u-})]. \end{aligned}$$

Follow the same exact analysis for $\sum_{0<u\le t} e^{-ru}[c(u,S_u)-c(u,S_{u-})]$ as in section (2.3) we have

$$\begin{aligned} e^{-rt}c(t,S_t) &= \int_0^t e^{-ru}\Big[-rc(u,S_u)+\frac{\partial}{\partial t}c(u,S_u)+\frac{\partial}{\partial x}c(t,S_u)(r-\widetilde{m})S_u \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial x^2}c(u,S_u)\sigma^2 S^2(u)+\sum_{i=1}^m [c(u,S_u(1+y_i))-c(u,S_u)]\widetilde{\lambda}_i\Big]du \\ &\quad +\int_0^t e^{-ru}\frac{\partial}{\partial x}c(t,S_u)S_u d\widetilde{W}(u)+\int_0^t e^{-ru}[c(u,S_u)-c(u,S_{u-})]dM(u), \end{aligned}$$

where

$$M(t)=\sum_{i=1}^m N_i(t)-\widetilde{\lambda}_i t$$

is a $\mathbb{Q}$-martingale.

Setting the dt part to be 0 gives the following:

Theorem 2.3. *The call option price $c(t,x)$ in the model of this section satisfies the differential difference equation*

$$\begin{aligned} -rc(t,x) &+ \frac{\partial}{\partial t}c(t,x)+(r-\widetilde{m})x\frac{\partial}{\partial x}c(t,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}c(t,x)\sigma^2x^2 \\ &+ \sum_{i=1}^m [c(t,x(1+y_i))-c(t,x)]\widetilde{\lambda}_i = 0, 0\le t<T, x>0; \\ c(T,x) &= (x-K)^+, x>0. \end{aligned}$$

2.5 A unifying approach via Levy measure

Note that all of the above derivations rely on the decomposition of a compound Poisson process with discrete jumps into sums of individual Poisson processes. This technique obviously does not work when we have a compound Poisson process with continuous jump distribution. The way to handle this situation is via the concept of the Levy measure. It will also help us write one single type of equation, called Partial Integro-Differential Equation (PIDE), for all types of our noise, as long as they are compound Poisson process plus a Brownian motion. For a more detailed treament of Levy process with application to finance, see e.g. [2].

2.5.1 The Levy measure

Definition 2.4. *Let L_t be a Levy process and $A \in \mathcal{B}(\mathbb{R})$ be a Borel measurable subset of the real line. Define*

$$\mu_t^L(A) := \sum_{0<s\leq t} \mathbf{1}_{\Delta L_s \in A};$$

that is, $\mu_t^L(A)$ counts the number of jumps of L, up to time t, that have size in the set A. We call $\mu_t^L(\cdot)$ the Poisson random measure associated with the Levy process L_t. Note that for a fixed A, $\mu_t^L(A)$ is a counting process. Also define

$$\nu(A) := \mathbb{E}(\mu_1^L(A)).$$

We say ν is the Levy measure associated with the Levy process L_t.

Remark 2.5. *In the above definition, usually one would require that the point 0 is "far away" from the set A, that is $0 \notin \overline{A}$. This is because a Levy process L_t can have infinitely many small jumps close to 0, which in turn may make $\mu_t^L(A)$ to be infinite if $0 \in \overline{A}$. However, in the cases we're dealing with, namely upto compound Poisson process, this will not happen. The number of jumps of compound Poisson process in any finite time interval $[0, t]$ will always remain finite. So we do not have to include this restriction in the set A, for ease of introduction to the material.*

Observe that for a fixed A, μ_t^L has independent and stationary increment, which is inherited from the Levy process L_t. Therefore, $\mu_t^L(A)$ is a Poisson process with rate

$$\lambda^A = \nu(A) = \mathbb{E}(\mu_1^L(A)).$$

In other words, the Levy measure ν measures *the expected number of jumps of L_t of a certain height in a time interval of length 1.* The height is determined by what values of the set A you plug in to the measure ν. We list what ν is for the processes we were familiar with in this chapter.

1. Poisson process with rate λ:

$$\nu(dx) = \lambda \delta_1(dx).$$

2. Compound Poisson process with rate λ and discrete jumps $y_1, \cdots, y_M$:

$$\nu(dx) = \lambda \sum_{m=1}^{M} p_m \delta_{y_m}(dx).$$

3. Compound Poisson process with rate λ and continuous jump distribution $f_Y(x)$:

$$\nu(dx) = \lambda f_Y(dx).$$

Remark 2.6. *Note that in all of the above examples,*

$$\lambda = \int_{\mathbb{R}} \nu(dx) = \nu(\mathbb{R}).$$

This indeed will be the case for all compound Poisson processes: they have finite Levy measure and the rate is equal to the Levy measure of the real line.

2.5.2 Integrating with respect to the random Poisson measure

For a Levy process L_t with Levy measure ν, adapted to a filtration $\mathcal{F}_t$. We define

$$\int_0^t \int_A f(s,x)\mu^L(ds,dx) := \sum_{0<s\leq t} f(s,\Delta L_s)\mathbf{1}_{\Delta L_s \in A}.$$

That is, the integral $\int_0^t \int_A f(s,x)\mu^L(ds,dx)$ is a pure jump process that jumps at the same time as L, with the jump size $f(s,\Delta L_s)$ if the jump of L happens at time s.

What will be important for us is the following martingale result:

Theorem 2.7. *Let $f(s,x,\omega)$ be a process with left continuous with right limit paths adapted to the filtration $\mathcal{F}_t$ satisfying certain integrability conditions. Then*

$$\int_0^t \int_A f(s,x,\omega)[\mu^L(ds,dx) - \nu(dx)ds]$$

is a $\mathcal{F}_t$-martingale.

Proof. The proof starts by approximating $f(t,x,\omega)$ by simple processes of the form $\sum_{k=1}^m \xi_k(t)\phi_k(x)$, where $\xi_k(t)$ are $\mathcal{F}_t$ measurable processes and ϕ_k are deterministic functions of x. We prove the martingale property for these simple processes and prove the general result by a convergence argument. For details see [1].

2.5.3 Ito's formula for jump processes, random Poisson measure version

Let X_t be a process of the form

$$X(t) = X_0 + \int_0^t \alpha(s)ds + \int_0^t \gamma(s)dWs + J(t),$$

where $J(t)$ is a compound Poisson process. Let f be a $C^{1,2}$ function. Then

$$\begin{aligned}
f(t, X(t)) &= f(0, X_0) + \int_0^t f_t(s, X_s)ds \\
&+ \int_0^t f_x(s, X_s)dX^c(s) + \int_0^t \frac{1}{2} f_{xx}(s, X_s)\gamma^2(s)ds \\
&+ \sum_{0<s\leq t} f(s, X_s) - f(s, X_{s-}) \\
&= f(0, X_0) + \int_0^t f_t(s, X_s)ds \\
&+ \int_0^t f_x(s, X_s)dX^c(s) + \int_0^t \frac{1}{2} f_{xx}(s, X_s)\gamma^2(s)ds \\
&+ \int_0^t \int_{\mathbb{R}} \Big[f(s, X_{s-} + x) - f(s, X_{s-}) \Big] \mu^J(ds, dx).
\end{aligned}$$

The reason for the re-writing in the random Poisson measure version is clear: we want to use the martingale result mentioned in the previous section. The equality

$$\sum_{0<s\leq t} f(s, X_s) - f(s, X_{s-}) = \int_0^t \int_{\mathbb{R}} \Big[f(s, X_{s-} + x) - f(s, X_{s-}) \Big] \mu^J(ds, dx)$$

comes from the fact that the jumps of X comes from the jumps of J, and $\Delta X_s = \Delta J_s$ at all jump times s.

2.5.4 PIDE for Euro call option with compound Poisson process and Brownian motion noise

Now suppose S_t has the dynamic:

$$dSt = rS_t dt + \sigma S_t d\widetilde{W}(t) + \gamma S_{t-} d(Q_t - \widetilde{\mu} t),$$

where we added a volatility component γ in the compound Poisson part for generality, even though this is not strictly necessary as it can be incoporated into the jumps of Q. This is the parameter σ in the previous sections (2.2), (2.3).

Recall that applying the Ito's formula, we have

$$\begin{aligned}
e^{-rt} c(t, S_t) &= \int_0^t e^{-ru} \Big[-rc(u, S_u) + \frac{\partial}{\partial t} c(u, S_u) + \frac{\partial}{\partial x} c(t, S_u)(r - \widetilde{m}) S_u \\
&+ \frac{1}{2} \frac{\partial^2}{\partial x^2} c(u, S_u) \sigma^2 S^2(u) \Big] du \\
&+ \int_0^t e^{-ru} \frac{\partial}{\partial x} c(t, S_u) S_u d\widetilde{W}(u) + \sum_{0<u\leq t} e^{-ru} [c(u, S_u) - c(u, S_{u-})].
\end{aligned}$$

Rewriting the term $\sum_{0<u\le t} e^{-ru}[c(u,S_u)-c(u,S_{u-})$ using random Poisson measure we have

$$\begin{aligned}\sum_{0<u\le t} e^{-ru}[c(u,S_u)-c(u,S_{u-}) &= \sum_{0<u\le t} e^{-ru}[c(u,S_{u-}(1+\gamma\Delta Q_t))-c(u,S_{u-})] \\ &= \int_0^t\int_{\mathbb{R}} e^{-ru}[c(u,S_{u-}(1+\gamma x))-c(u,S_{u-})]\mu^Q(du,dx).\end{aligned}$$

Thus applying the martingale result, we have

$$\begin{aligned}e^{-rt}c(t,S_t) &= \int_0^t e^{-ru}\Big[-rc(u,S_u)+\frac{\partial}{\partial t}c(u,S_u)+\frac{\partial}{\partial x}c(t,S_u)(r-\widetilde{m})S_u \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial x^2}c(u,S_u)\sigma^2S^2(u)\Big]du+\int_0^t e^{-ru} \\ &\quad \frac{\partial}{\partial x}c(t,S_u)S_u d\widetilde{W}(u)+\int_0^t\int_{\mathbb{R}} e^{-ru}[c(u,S_{u-}(1+\gamma x))-c(u,S_{u-})]\mu^Q(du,dx) \\ &= \int_0^t e^{-ru}\Big[-rc(u,S_u)+\frac{\partial}{\partial t}c(u,S_u)+\frac{\partial}{\partial x}c(t,S_u)(r-\widetilde{m})S_u \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial x^2}c(u,S_u)\sigma^2S^2(u)+\int_{\mathbb{R}}[c(u,S_{u-}(1+\gamma x))-c(u,S_{u-})]\nu(dx)\Big]du \\ &\quad +\int_0^t e^{-ru}\frac{\partial}{\partial x}c(t,S_u)S_u d\widetilde{W}(u) \\ &\quad +\int_0^t\int_{\mathbb{R}} e^{-ru}[c(u,S_{u-}(1+\gamma x))-c(u,S_{u-})](\mu^Q(du,dx)-\nu(dx)du).\end{aligned}$$

Therefore, $c(t,x)$ satisfies the PIDE

$$\begin{aligned}-rc(t,x) &+ \frac{\partial}{\partial t}c(t,x)+(r-\widetilde{m})x\frac{\partial}{\partial x}c(t,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}c(t,x)\sigma^2x^2 \\ &+ \int_{\mathbb{R}}[c(t,x(1+\gamma z))-c(t,x)]\nu(dz)=0, 0\le t<T, x>0; \\ c(T,x) &= (x-K)^+, x>0.\end{aligned}$$

In particular we have:
(i) If Q is a Poisson $(\widetilde{\lambda})$ process then $\nu(dz)=\widetilde{\lambda}\delta_1(dz)$. Thus the PIDE becomes

$$\begin{aligned}-rc(t,x) &+ \frac{\partial}{\partial t}c(t,x)+(r-\widetilde{m})x\frac{\partial}{\partial x}c(t,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}c(t,x)\sigma^2x^2 \\ &+ \widetilde{\lambda}[c(t,x(1+\gamma))-c(t,x)]=0.\end{aligned}$$

(ii) If Q is a compound Poisson with discrete jumps then $\nu(dz) = \sum_{m=1}^{M} \widetilde{\lambda}_m \delta_{ym}(dz)$. Thus the PIDE becomes

$$\begin{aligned} -rc(t,x) &+ \frac{\partial}{\partial t}c(t,x) + (r - \widetilde{m})x\frac{\partial}{\partial x}c(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}c(t,x)\sigma^2 x^2 \\ &+ \sum_{m=1}^{M} \widetilde{\lambda}_m [c(t, x(1 + \gamma y_m)) - c(t,x)] = 0. \end{aligned}$$

(iii) If Q is a compound Poisson with continuous jump then $\nu(dz) = \lambda f(z)dz$. Thus the PIDE becomes

$$\begin{aligned} -rc(t,x) &+ \frac{\partial}{\partial t}c(t,x) + (r - \widetilde{m})x\frac{\partial}{\partial x}c(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}c(t,x)\sigma^2 x^2 \\ &+ \int_{\mathbb{R}} [c(t, x(1 + \gamma z) - c(t,x)]\lambda f(z)dz = 0. \end{aligned}$$

You should verify that for cases (i) and (ii) the results are exactly as what we got before in sections (2.2), (2.3).

References

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[2] Papapantoleon, Antonis. *An introduction to Lé vy processes with applications in finance.* arXiv preprint arXiv:0804.0482 (2008).