Change of measure for jump processes

Math 622 - Spring 2015

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1 Motivation

One of the fundamental concept in Math Finance I regarding the Black-Scholes model is the following: Suppose $S(t)$ satisfies

$$S(t) = S(0) + \int_0^t \mu(u)S(u)du + \int_0^t S(u)dW(u),$$

under the objective probability $\mathbb{P}$. Unless $\mu(u) = r$, the interest rate, (which we supposed to be a constant for simplicty) $e^{-rt}S(t)$ *is not a martingale under* $\mathbb{P}$, and thus we cannot price financial product under $\mathbb{P}$. We need to find another measure $\mathbb{Q}$, the risk neutral measure, so that $e^{-rt}S(t)$ is a martingale udner $\mathbb{Q}$. The key idea is that under $\mathbb{Q}$, it must be the case that $\widetilde{W}(t) := \int_0^t (\mu(u) - r)du + W(t)$ is a Brownian motion. So that

$$S(t) = S(0) + \int_0^t rS(u)du + \int_0^t S(u)d\widetilde{W}(u)$$

has the right distribution under $\mathbb{Q}$.

Intuitively, the measure $\mathbb{Q}$ is chosen so that we can "modify the drift" of $W(t)$ and still have the new process $\widetilde{W}(t)$ being a Brownian motion; which results in modifying the drift of $S(t)$ to the desirable drift(in this case, r).

Now suppose $S(t)$ satisfies

$$S(t) = S(0) + \int_0^t \mu S(u)du + \int_0^t S(u-)dM(u),$$

under some objective probability measure $\mathbb{P}$, where $M(t) = N(t) - \lambda t$ is a compensated Poisson process with rate λ under $\mathbb{P}$. Again, we would like that

$$S(t) = S(0) + \int_0^t rS(u)du + \int_0^t S(u-)d\widetilde{M}(u),$$

where $\widetilde{M}(t) := M(t) - (r-\mu)t$ is a martingale under a probability measure $\mathbb{Q}$. Again, since $M(t) = N(t) - \lambda t$, it is clear that $\widetilde{M}(t)$ is a martingale if $N(t)$ becomes a Poisson process with rate $\lambda + (r-\mu)$ under $\mathbb{Q}$. This note discusses how to choose such a measure $\mathbb{Q}$ for various choices of jump martingales M.

2 Review of change of measure, Girsanov's theorem

2.1 The change of measure kernel $Z(t)$

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$; $\mathcal{F}(t), 0 \leq t \leq T$ a filtration with $\mathcal{F}(T) = \mathcal{F}$. If we define another probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}(T))$ via the relation

$$d\mathbb{Q} = Z(T) d\mathbb{P},$$

for some random variable $Z(T)$, that is for all $Y \in \mathcal{F}(T)$

$$\mathbb{E}^{\mathbb{Q}}(Y) := \mathbb{E}^{\mathbb{P}}(Z(T)Y),$$

it must be that $\mathbb{P}(Z(T) \geq 0) = 1$ *and* $\mathbb{E}^{\mathbb{P}}(Z(T)) = 1$.

2.2 Restriction of $\mathbb{Q}$ to a smaller sigma algebra $\mathcal{F}(t)$

Let $\mathcal{F}(t), 0 \leq t \leq T$ be a filtration associated with a probability space $(\Omega, \mathbb{P}, \mathcal{F}(T))$. If $Z(t)$ is *a* $\mathbb{P}$ *martingale*, $Z(T)$ satisfies the conditions in (i), then for all $Y \in \mathcal{F}(t)$

$$\mathbb{E}^{\mathbb{Q}}(Y) = \mathbb{E}^{\mathbb{P}}(Z(t)Y).$$

(See Shreve's Lemma 5.2.1.) Note that this is *not a definition* but a result that follows from the definition in (i) and the fact that Z is a martingale.

2.3 Conditional expectation in change of measure

Let Y be $\mathcal{F}_T$ measureable, we have for $t \leq T$

$$\mathbb{E}^{\mathbb{Q}}(Y|\mathcal{F}(t)) = \frac{\mathbb{E}^{\mathbb{P}}(Z_T Y|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(Z_T|\mathcal{F}_t)}.$$

In particular, if $Z(t)$ *is a* $\mathbb{P}$*-martingale* then combining the results above we have for $s \leq t$ and $Y \in \mathcal{F}_t$

$$\mathbb{E}^{\mathbb{Q}}(Y|\mathcal{F}_s) = \frac{\mathbb{E}^{\mathbb{P}}(Z_t Y|\mathcal{F}_s)}{Z_s}.$$

2.4 Condition for a process to be a martingale under the new measure

Theorem 2.1. *Let $X(t)$ be a $\mathcal{F}(t)$ adapted process, $Z(t)$ a $\mathbb{P}$-martingale then $X(t)Z(t)$ is a $\mathbb{P}$ martingale* **if and only if** *$X(t)$ is a $\mathbb{Q}$-martingale.*

Application: Let $X(t)$ be the "Brownian motion with drift" $\widetilde{W}(t)$ or the process $\widetilde{M}(t)$ in section I. Recall that we want $\widetilde{W}(t)$ (or $\widetilde{M}(t)$) be a martingale under $\mathbb{Q}$. This statement gives a sufficient condition for this to happen.

3 Some remarks about Girsanov theorem

3.1 Characterization of Brownian motion

We all know the Levy's characterization of Brownian motion: continuous martingale with quadratic variation on $[0, t]$ equals to t. There is an equivalent characterization:

Theorem 3.1. *Let $X(t)$ be a continuous process such that $X(0) = 0$. Then $X(t)$ is a Brownian motion w.r.t a filtration $\mathcal{F}(t)$ if and only if for all $u \in \mathbb{R}$,*

$$\mathcal{E}^u(X)(t) := e^{uX_t - \frac{1}{2}u^2 t}$$

is a martingale w.r.t $\mathcal{F}_t$.

Proof. Let $X(t)$ be a Brownian motion. It is routine to show that

$$e^{uX(t) - \frac{1}{2}u^2 t}$$

is a martingale.

The converse can be argued heuristically as followed. Suppose that $e^{uX_t - \frac{1}{2}u^2 t}$ is a martingale for all $u \in \mathbb{R}$. Then by definition for $s < t$

$$\mathbb{E}(e^{u(X_t - X_s)} \Big| \mathcal{F}_s) = e^{\frac{1}{2}u^2(t-s)}.$$

Since *for all* $u \in \mathbb{R}$, the RHS is independent of $\mathcal{F}(s)$, $X(t) - X(s)$ is independent of $\mathcal{F}(s)$ (see explaination in the remark below). Hence it has **independent increments**.

Moreover, from the same calculation, the moment generating function of $X(t) - X(s)$ is that of a Normal(0, $t - s$). Hence it has **stationary increments**, and the increments has Normal(0, $t - s$) distribution. Thus $X(t)$ is a Brownian motion.

Remark 3.2. *We clarifiy the reason why $X(t) - X(s)$ is independent of $\mathcal{F}(s)$. We make the following claim: if $E(e^{uX}|\mathcal{F}) = E(e^{uX})$ for all $u \in \mathbb{R}$ then X is independent of $\mathcal{F}$.*

This claim is true in turn because of the following result, known as Kac's theorem for characteristic functions: if

$$E(e^{uX+vY}) = E(e^{uX})E(e^{vY}), \forall u, v \in \mathbb{R},$$

then X, Y are independent. A reference can be found in Thereom 1.1.16 of the textbook Levy's processes and Stochastic Calculus by David Applebaum.

If we accept this result, then we see that or all $Y \in \mathcal{F}$,

$$E(e^{uX+vY}) = E\big(E(e^{uX+vY}|\mathcal{F})\big) = E\big(e^{vY}E(e^{uX}|\mathcal{F})\big) = E(e^{uX})E(e^{vY}).$$

Hence X is indepedent of Y for all $Y \in \mathcal{F}$. Hence X is independent of $\mathcal{F}$.

3.2 Choice of $Z(t)$ in Girsanov Theorem

Suppose that $W(t)$ is a $\mathbb{P}$ Brownian motion. We want to find $\mathbb{Q}$ via

$$d\mathbb{Q} = Z(T)d\mathbb{P}$$

so that

$$\widetilde{W}(t) := W(t) + \alpha t$$

is a $\mathbb{Q}-$Brownian motion. From the characterization of Brownian motion from the exponential martingale above, we need

$$\mathcal{E}^u(\widetilde{W})(t) = e^{u\widetilde{W}_t - \frac{1}{2}u^2 t}$$

to be a $\mathbb{Q}$ martingale. Observe that

$$\mathcal{E}^u(\widetilde{W})(t) = e^{u\widetilde{W}(t) - \frac{1}{2}u^2 t} = e^{uW(t) - \frac{1}{2}\left(u^2 - 2u\alpha\right)t}.$$

By Theorem (2.1), in order for $\mathcal{E}^u(\widetilde{W})(t)$ to be a $\mathbb{Q}$ martingale, we need to choose the change of measure kernel $Z(t)$ so that both $Z(t)$ **and** $\mathcal{E}^u(\widetilde{W})(t)Z(t)$ are $\mathbb{P}$-martingales. Since

$$u^2 - 2u\alpha = (u - \alpha)^2 - \alpha^2,$$

and clearly

$$e^{(u-\alpha)W(t)-\frac{1}{2}(u-\alpha)^2t}$$

is a $\mathbb{P}$-martingale, we may guess the choice for $Z(t)$ is

$$Z(t) = e^{-\alpha W(t)-\frac{1}{2}\alpha^2 t},$$

which is clearly also a $\mathbb{P}$-martingale.

This intuition also suggests that if we want $\widetilde{W}(t) = W(t) + \int_0^t \alpha(u)du$ to be a $\mathbb{Q}$ Brownian motion, the choice of $Z(t)$ is

$$Z(t) = e^{-\int_0^t \alpha(u)dW(u)-\frac{1}{2}\int_0^t \alpha(u)^2du},$$

even though the verification now is slightly more involved.

4 Change of measure for Poisson processes

4.1 Poisson process characterization

Theorem 4.1. *A càdlàg process $N(t)$, $N(0) = 0$, is a Poisson process with rate λ w.r.t $\mathcal{F}(t)$ if and only if for all $u \in \mathbb{R}$*

$$\exp\big(uN(t) - \lambda t(e^u - 1)\big)$$

is a martingale w.r.t $\mathcal{F}(t)$.

The proof for this theorem is similar to the proof for the characterization of Brownian motion.

4.2 Choice of $Z(t)$

Suppose $N(t)$ is a Poisson process with rate λ under $\mathbb{P}$. We want to find $\mathbb{Q}$ via the change of measure formula

$$d\mathbb{Q} = Z(T)d\mathbb{P}$$

so that $N(t)$ has rate $\widetilde{\lambda}$ under $\mathbb{Q}$. By the characterization of Poisson process, we want

$$\exp\big(uN(t) - \widetilde{\lambda}t(e^u - 1)\big)$$

to be a $\mathbb{Q}$-martingale.

Again, by Theorem (2.1), we need to choose $Z(t)$ so that both

$$Z(t) \text{ and } e^{uN(t)-\widetilde{\lambda}t(e^u-1)}Z(t)$$

are $\mathbb{P}$-martingales.

Now the choice of such $Z(t)$ may not be immediately obvious, even though we can use the same reverse engineer idea as we did above with the Brownian motion. Instead, a more natural idea here is to choose a *general exponential martingale* $Z(t)$ associated with $N(t)$ with *an undetermined coefficient.* We perform the change of measure with $Z(t)$ and *expect* that $N(t)$ will remain a Poisson process with different rate under this change of measure. We then observe what rate $N(t)$ will actually be under the new measure using the exponential martingale characterization of a Poisson process. Then we can determine the precise coefficient to achieve the desired rate of $N(t)$ under the new measure.

More specifically, we let

$$Z(t) = e^{aN(t)-\lambda t(e^a-1)},$$

where a is our undetermined coefficient. Then clearly $Z(t)$ is a $\mathbb{P}$-martingale.

Suppose that $N(t)$ remains a Poisson process under $\mathbb{Q}$ and its rate is $\widetilde{\lambda}$. Then by Theorem (2.1) and the exponential martingale characterization of Poisson processes we must have

$$e^{uN(t)-\widetilde{\lambda}t(e^u-1)}e^{aN(t)-\lambda t(e^a-1)} = e^{(u+a)N(t)-\widetilde{\lambda}t(e^u-1)-\lambda t(e^a-1)}$$

is a $\mathbb{P}$-martingale.

But since we know

$$e^{(u+a)N(t)-\lambda t(e^{u+a}-1)}$$

is a $\mathbb{P}$-martingale we must have

$$\widetilde{\lambda}(e^u-1)+\lambda(e^a-1) = \lambda(e^{u+a}-1). \tag{1}$$

Note that the above equation has to be true $\forall u \in \mathbb{R}$. In particular if we choose $u=-a$ then the RHS equals 0. Thus

$$\widetilde{\lambda} = \lambda\frac{1-e^a}{e^{-a}-1} = \lambda\frac{e^a(1-e^a)}{1-e^a} = \lambda e^a.$$

Plug this in we indeed verify the equation (1) for all u.

Thus our conclusion is that if we choose

$$Z(t) = e^{aN(t)-\lambda t(e^a-1)},$$

then $N(t)$ is a Poisson process with rate λe^a under $\mathbb{Q}$. Now if we desire

$$\lambda e^a = \tilde{\lambda},$$

for some pre-given $\tilde{\lambda}$ then clearly $a = \log\left(\frac{\tilde{\lambda}}{\lambda}\right)$ and also

$$Z(t) = \exp\left[\log\left(\frac{\widetilde{\lambda}}{\lambda}\right) N(t) + (\lambda - \widetilde{\lambda})t\right].$$

We have arrived at the following theorem

Theorem 4.2. *Let $N(t)$ be a Poisson process with rate λ under a probabilty $\mathbb{P}$ and $\mathcal{F}(t)$ a filtration for $N(t)$. Let $\widetilde{\lambda}$ be given. Define*

$$\begin{aligned} Z(t) &:= \exp\left[\log\left(\frac{\widetilde{\lambda}}{\lambda}\right) N(t) + (\lambda - \widetilde{\lambda})t\right] \\ &= e^{(\lambda-\widetilde{\lambda})t}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N(t)}, 0 \le t \le T. \end{aligned}$$

Also define

$$d\mathbb{Q} = Z(T)d\mathbb{P} \text{ on } \mathcal{F}(T).$$

Then $Z(t)$ is a $\mathbb{P}$ martingale and under $\mathbb{Q}$, $N(t)$ is a Poisson process with rate $\widetilde{\lambda}$.

5 Change of measure for compound Poisson with discrete jump distribution

Let $Q(t)$ be a compound Poisson process with rate λ. That is

$$Q(t) = \sum_{i=1}^{N(t)} Y_i,$$

$N(t)$ has rate λ under a probabilty $\mathbb{P}$ and $\mathcal{F}(t)$ a filtration for $Q(t)$. Recall that each jump of $Q(t)$ has identical distribution Y_i.

Here we assume that Y_1 (hence all Y_i's) takes values $y_1, y_2, ..., y_M$ with probability

$$\mathbb{P}(Y_1 = y_m) = p_m, 1 \leq m \leq M$$

that is Y_1 has discrete distribution.

We want to change the intensity of $Q(t)$ as well as the distribution of Y_i (that is to change p_m) via the change of measure. For any $\widetilde{\lambda} > 0$ and $\widetilde{p}_m \in (0, 1), \sum_{m=1}^{M} \widetilde{p}_m = 1$ we find a probabilty $\mathbb{Q}$ so that under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\widetilde{\lambda}$ and Y_i has distribution

$$\mathbb{Q}(Y_1 = y_m) = \widetilde{p}_m, 1 \leq m \leq M.$$

Before we proceed, we need to mention an important result about decomposing a compound Poisson process with discrete jumps into a sum of Poisson processes.

5.1 Summing and Decomposing Compound Poisson processes

5.1.1 Summing compound Poisson processes

Compound Poisson processes can be combined and decomposed in fascinating ways. Shreve treats these in Theorem 11.3.3, page 471, and Corollary 11.3.3, page 473, for the special case when $Y_1, \ldots$ are discrete random variables. We will state more general versions of these properties here, but without a proof: Shreve gives a proof for his special case.

Theorem 11.3.3 says in essence that one can build a compound Poisson process by bringing in jumps of different sizes at different Poisson rates. For example let N_1 and N_2 be two independent, Poisson processes with respective rates λ_1 and λ_2. Then $y_1 N_1(t)$ is a very simple compound Poisson process in which jumps of size y_1 arrive in a Poisson stream of rate λ_1, and $y_2 N_1(t)$ is a very simple compound Poisson process in which jumps of size y_2 arrive in a Poisson stream of rate λ_2.

Let

$$Q(t) = y_1 N_1(t) + y_2 N_2(t).$$

This is a pure jump process whose jumps are either of size y_1 or y_2. The total number of jumps by time t is clearly $N_1(t) + N_2(t)$. Let Y_k denote the size of the kth jump of Q. Then, by definition,

$$Q(t) = \sum_{k=1}^{N_1(t)+N_2(t)} Y_k.$$

Then Theorem 11.3.3 says

(i) $N_1 + N_2$ is a Poisson process with rate $\lambda_1 + \lambda_2$;

(ii) $Y_1, Y_2, \dots$ are independent and identically distributed with

$$I\!P\Big(Y_i = y_1\Big) = \frac{\lambda_1}{\lambda_1+\lambda_2} \quad \text{and} \quad I\!P\Big(Y_i = y_2\Big) = \frac{\lambda_2}{\lambda_1+\lambda_2}.$$

As a consequence, Q is a compound Poisson process. Theorem 11.3.3 extends this idea to summing Poisson streams of more than two possible jump sizes. Note that, as a consequence of statement (i) above, *the independent Poisson processes N_1 and N_2 never jump at the same time.*

Theorem 11.3.3 is actually a special case of a much more general theorem: the sum of any finite number of compound Poisson processes is a compound Poisson process.

We give a heuristic reasoning about Theorem 11.3.3. First observe that if $N(t) = N_1(t) + N_2(t)$ then $N(t)$ would jump at the jump time of N_1 or N_2, whichever arrives first. That is $N(t)$ jumps at the minimum of the jump times of N_1 and N_2. Now let $\tau_i, i = 1, 2$ are independent exponential(λ_i) random variables and $\tau = \min(\tau_1, \tau_2)$ then

$$P(\tau \geq t) = P(\tau_1 \geq t)P(\tau_2 \geq t) = e^{-(\lambda_1+\lambda_2)t}.$$

That is τ is an exponential$(\lambda_1 + \lambda_2)$ random variable. This gives the intuition about $N_1 + N_2$ being a Poisson process with rate $\lambda_1 + \lambda_2$. The rigorous proof would use the exponential martingale characterization of Poisson processes mentioned above.

Second, $N(t)$ would jump with size y_1 if the jump time of N_1 arrives before the jump time of N_2 and vice versa. We have

$$\begin{aligned} P(\tau_1 < \tau_2) &= \int_0^\infty P(\tau_1 < \tau_2 | \tau_2 = t)\lambda_2 e^{-\lambda_2 t} dt \\ &= \int_0^\infty (1 - e^{-\lambda_1 t})\lambda_2 e^{-\lambda_2 t} dt \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2}. \end{aligned}$$

Similarly

$$P(\tau_1 < \tau_2) = \frac{\lambda_2}{\lambda_1+\lambda_2}.$$

This explains the distribution of Y_i, the jump size of N_t.

5.1.2 Decomposing compound Poisson processes

One can also go in the opposite direction and decompose a compound Poisson process into a sum of independent Poisson processes bringing different sized jumps at different rates. We shall state this very generally. In fact, we will start not with a compound Poisson process, but just with some Lévy process.

Let X be a Lévy process. Let A be a subset of $I\!R$ that avoids a neighborhood of 0 in the sense that for some $\epsilon > 0$, $A \bigcap (-\epsilon, \epsilon) = \emptyset$. Let

$$N_A(t) := \sum_{s \le t} \mathbf{1}_{\{\triangle X(s) \in A\}}$$

be the number of jumps of X with values in A that occur by time t. Let

$$X_A(t) := \sum_{s \le t} \triangle X(s) \mathbf{1}_{\{\triangle X(s) \in A\}},$$

$X_A(t)$ will be well defined whenever $N_A(t)$ is finite. The process X_A is the accumulated sum of all the jumps of X with values in the set A. It might be that X never has a jump with values in A (that is, $N_A(t) = 0$ for all $t \ge 0$). But if it does, let $Y_1^A, Y_2^A, \ldots,$ be the first, second, third, etc. jump values of $\{X_A(t);\, t \ge 0\}$. By definition of the terms so far,

$$X_A(t) = \sum_{k=1}^{N_A(t)} Y_k^A.$$

Theorem 1. *(a) Let $X(t)$ be a Lévy process. Then $\{X_A(t);\, t \ge 0\}$ is a compound Poisson process; in other words, N_A is a Poisson process and $Y_1^A, Y_2^A, \ldots$ is a sequence of independent, identically distributed random variables independent of N_A. In addition, $X(t) - X_A(t)$ is a Lévy process and is independent of $X_A(t)$. Thus, $X(t) = X(t) - X_A(t) + X_A(t)$ represents X as the sum of two independent Lévy processes, the first of which has no jumps with values in A, and the second of which only has jumps with values in A.*

(b) Let $\epsilon > 0$ and let $A_1, \ldots, A_n$ be disjoint subsets of $(-\infty, \infty) - (-\epsilon, \epsilon)$. Then $X_{A_1}(\cdot), \ldots, X_{A_n}(\cdot)$ are independent *compound Poisson processes that are all independent of $X(\cdot) - \Big(X_{A_1} + \cdots + X_{A_n}(\cdot)\Big)$.*

The take-home message of this theorem is that the accumulated jumps of a Lévy process into disjoint sets bounded away from 0 are independent, compound Poisson processes. Thus, a Lévy process with jumps has a very rich structure which aggregates the influence of many, independently occurring Poisson streams. Corollary 11.3.4 is a special case of this theorem for compound Poisson process that admit only a finite number of possible jumps sizes.

Theorem 1 is at the heart of a result, due to Lèvy and Khinchine, that characterizes the most general Lèvy process. It says that if X is a Lèvy process, there is a decomposition of the form $X(t) = \mu t + \sigma W(t) + Y(t)$, where W is a Brownian motion independent independent of Y, and where Y is a limit of a sequence of processes $Z_n(t) + m_n t$, with Z_n being a compound Poisson process for each n.

Lastly we give a heuristic reasoning for decomposing a compound Poisson process with discrete jumps. Let

$$Q(t) = \sum_{i=0}^{N(t)} Y_i,$$

where $N(t)$ is a Poisson(λ) process and $P(Y_1 = y_m) = p_m, m = 1, \cdots, M$.

Now if we let

$$Q_m(t) = \sum_{i=0}^{N(t)} Y_i \mathbf{1}_{\{Y_i = y_m\}},$$

then observe that $Q_m(t)$ has independent and stationary increments. That is it is a Levy process. Moreover, one can check that

$$Ee^{u\frac{Q_m(t)}{y_m}} = e^{\lambda p_m t(e^u - 1)}.$$

That is $N_m(t) := \frac{Q_m(t)}{y_m}$ is a Poisson (λp_m) process. Lastly, it is clear from the definition that for $n \neq m$, $N_m(t)$ and $N_n(t)$ *do not jump at the same time.* From an exercise in Homework 3, you'll see that this implies $N_m(t)$ and $N_n(t)$ are independent. This gives the decomposition of $Q(t)$ as

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t),$$

where $N_m(t)$ are independent Poisson processes with rates λp_m.

5.2 Change of measure for multiple independent Poisson processes

Lemma 5.1. *Let $N_m, m = 1, \cdots, M$ be independent Poisson processes with rates $\lambda_m, m = 1, \cdots, M$. Let $\widetilde{\lambda}_m, m = 1, 2, ..., M$ be given. Define*

$$\begin{aligned} Z_m(t) &:= e^{(\lambda_m - \widetilde{\lambda}_m)t} \left(\frac{\widetilde{\lambda}_m}{\lambda_m} \right)^{N_m(t)} \\ Z(t) &:= \prod_{m=1}^{M} Z_m(t) \end{aligned}$$

and

$$d\mathbb{Q} = Z(T) d\mathbb{P} \text{ on } \mathcal{F}(T).$$

Then N_m's are independent Poisson processes with rate $\widetilde{\lambda}_i$ under $\mathbb{Q}$.

You will be asked to explore the proof of this Lemma in Homework 2 for the case $M = 2$. The proof for general M is similar.

5.3 Change of measure for compound Poisson process with discrete jumps

The decomposition of a compound Poisson process into multiple independent Poisson processes and Lemma (5.1) lead to the following result: (Shreve's Lemma 11.6.4, Theorem 11.6.5)

Theorem 5.2. *Let*

$$Q(t) = \sum_{i=1}^{N(t)} Y_i,$$

$N(t)$ has rate λ under a probabilty $\mathbb{P}$ and Y_1 takes values $y_1, y_2, ..., y_M$ with probability

$$\mathbb{P}(Y_1 = y_m) = p_m, 1 \leq m \leq M$$

that is Y_1 has discrete distribution.

Let $\widetilde{\lambda}_m, m = 1, 2, ..., M$ be given. Define $Z(t)$ as in Lemma (5.1). That is

$$Z(t) := \prod_{m=1}^{M} e^{(\lambda_m - \widetilde{\lambda}_m)t} \left(\frac{\widetilde{\lambda}_m}{\lambda_m} \right)^{N_m(t)}.$$

Then Z(t) is a $\mathbb{P}$ martingale. Moreover, under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\widetilde{\lambda}$ and $P^Q(Y_i = y_m) = \widetilde{p}_m$, where

$$\begin{aligned} \widetilde{\lambda} &= \sum_{m=1}^{M} \widetilde{\lambda}_m \\ \widetilde{p}_m &= \frac{\widetilde{\lambda}_m}{\widetilde{\lambda}}. \end{aligned}$$

Remark 5.3. *We mentioned at the beginning of this section that we can choose $\widetilde{\lambda}$ and $\widetilde{p}_m$, while Theorem (5.2) says we can choose $\widetilde{\lambda}_m$. The difference is artificial. Indeed, given $\widetilde{\lambda}_m$ we can define $\widetilde{\lambda}$ and $\widetilde{p}_m$ as in Theorem (5.2). But conversely, we can start out with $\widetilde{\lambda}$ and $\widetilde{p}_m$ and define $\widetilde{\lambda}_m := \widetilde{p}_m\widetilde{\lambda}$. It's up to you and the problem you're dealing with to decide which are the given variables to work with.*

5.4 Compound Poisson with continuous jump distribution

Let $Q(t)$ be a compound Poisson process with rate λ under a probabilty $\mathbb{P}$ and $\mathcal{F}(t)$ a filtration for $Q(t)$. Here we assume Y_i has continuous distribution with density function f.

We want to change the intensity of $Q(t)$ as well as the distribution of Y_i (that is the density f) via the change of measure. For any density function $\widetilde{f}$ and $\widetilde{\lambda}$, we find a probabilty $\mathbb{Q}$ so that under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\widetilde{\lambda}$ and Y_i has continuous distribution with densitry $\widetilde{f}$.

5.5 A rewrite of $Z(t)$ in Theorem (5.2)

There is yet another way to write the process $Z(t)$ in Theorem (5.2). Note that

$$\begin{aligned} Z(t) &= \prod_{m=1}^{M} e^{(\lambda_m - \widetilde{\lambda}_m)t} \left(\frac{\widetilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)} \\ &= e^{(\lambda-\widetilde{\lambda})t} \prod_{m=1}^{M} \left(\frac{\widetilde{\lambda}\widetilde{p}_m}{\lambda p_m}\right)^{N_m(t)} \\ &= e^{(\lambda-\widetilde{\lambda})t} \frac{\widetilde{\lambda}^{N(t)} \prod_{m=1}^{M} \widetilde{p}_m^{N_m(t)}}{\lambda^{N(t)} \prod_{m=1}^{M} p_m^{N_m(t)}}. \end{aligned}$$

By rearranging terms,

$$\prod_{m=1}^{M} p_m^{N_m(t)} = \prod_{i=1}^{N(t)} p(Y_i),$$

where we define

$$p(Y_i) := p_m \text{ if } Y_i = y_m, m = 1, \cdots, M.$$

To see this equality, note that for each m, there are $N_m(t)$ terms of p_m on the LHS. By definition, for each event ω, the Y_i random variables take on values y_m exactly $N_m(t)$ times. Thus there are also $N_m(t)$ terms of p_m on the RHS.

Similarly we have,

$$\prod_{m=1}^{M} \widetilde{p}_m^{N_m(t)} = \prod_{i=1}^{N(t)} \widetilde{p}(Y_i).$$

Thus

$$\begin{aligned} Z(t) &= e^{(\lambda-\widetilde{\lambda})t} \frac{\widetilde{\lambda}^{N(t)} \prod_{i=1}^{N(t)} \widetilde{p}(Y_i)}{\lambda^{N(t)} \prod_{i=1}^{N(t)} p(Y_i)} \\ &= e^{(\lambda-\widetilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\widetilde{\lambda}\widetilde{p}(Y_i)}{\lambda p(Y_i)}. \end{aligned}$$

5.6 Change of measure for compound Poisson with continuous jump distribution

The above observation suggests the following choice of $Z(t)$ when Y_i has continuous distribution.

Definition 5.4. *Fix $T > 0$. Let $\widetilde{\lambda} > 0$ and a density function $\widetilde{f}$ be given. Define*

$$Z(t) := e^{(\lambda-\widetilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\widetilde{\lambda}\widetilde{f}(Y_i)}{\lambda f(Y_i)}. \tag{2}$$

Also define

$$d\mathbb{Q} = Z(T)d\mathbb{P} \text{ on } \mathcal{F}(T).$$

Remark 5.5. *Since the density function f can be 0, to avoid dividing by 0, we assume $\widetilde{f}(y) = 0$ whenever $f(y) = 0$.*

We have the important results: (Shreve's Lemma 11.6.6, Theorem 11.6.7)

Theorem 5.6. *Z(t) defined in* (2) *is a* $\mathbb{P}$ *martingale (w.r.t.* $\mathcal{F}(t)$*). Under* $\mathbb{Q}$*,* $Q(t)$ *is a compound Poisson process with rate* $\widetilde{\lambda}$ *and* Y_i *has continuous distribution with density* $\widetilde{f}$*.*

Proof.

The proof of this Theorem relies on the following exponential martingale characterization of a compound Poisson process:

Let $\phi(u) := \mathbb{E}(e^{uY})$ be the moment generating function of a random variable Y. Then $Q(t)$ is a compound Poisson process with jump rate λ and i.i.d. jump size Y_i with moment generating function $\phi(u)$ if and only if

$$Z(t) = \exp\big(uQ(t) - \lambda t(\phi(u) - 1)\big)$$

is a martingale $\forall u \in \mathbb{R}$.

The details are left to the readers.

6 Change of measure for compound Poisson process and Brownian motion

We now consider the case when we have both a compound Poisson process $Q(t)$ and a Brownian motion $W(t)$. We want to find a change of measure kernel $Z(t)$ that would change the rate and the jump distribution of $Q(t)$ and the drift of $W(t)$. First we discuss an easier case when $Q(t)$ is just a Poisson process.

6.1 Change of measure for Poisson process and Brownian motion

We first describe an exponential martingale characterization result for Poisson process and Brownian motion.

Lemma 6.1. *$N(t)$ is a Poisson process with rate λ and $W(t)$ is a Brownian motion adapted to a filtration $\mathcal{F}(t)$ and they are independent if and only if*

$$e^{u_1 W_t - \frac{1}{2}u_1^2 t + u_2 N_t - \lambda t(e^{u_2} - 1)}$$

is a $\mathcal{F}(t)$-martingale for all $u_1, u_2 \in \mathbb{R}$.

The proof of the Lemma follows a similar idea as the proof of the exponential martingale characterization of a Brownian motion or a Poisson process described above. It is clear that when the martingale condition holds then W_t is a Brownian Motion and $N(t)$ is a Poisson process since we can choose $u_1 = 0$ or $u_2 = 0$. The independence follows from the Kac's theorem for characteristic function mentioned in (3.2) since the martingale condition being true also implies that

$$E\Big(e^{u_1 W_t + u_2 N_t}\Big) = E\Big(e^{u_1 W_t}\Big) E\Big(e^{u_2 N_t}\Big), \forall u_1, u_2 \in \mathbb{R}.$$

An interesting thing to note is that if $W(t)$ is a Brownian motion and $N(t)$ is a Poisson process adapted to the same filtration $\mathcal{F}(t)$ then they are **automatically** independent. To see this, apply Ito's formula to $e^{u_1 W_t - \frac{1}{2}u_1^2 t + u_2 N_t - \lambda t(e^{u_2}-1)}$ to conclude that it is a martingale. Then we can invoke Kac's theorem to show independence.

With the above characterization, the following change of measure result is automatic upon our previous discussion on the change of measure for Brownian motion and Poisson process.

Theorem 6.2. *Let $N(t)$ be a Poisson process with rate λ and $W(t)$ is a Brownian motion under $\mathbb{P}$. Let $\widetilde{\lambda} > 0$ be given. Define*

$$\begin{aligned} Z_1(t) &:= \exp\Big[- \int_0^t \theta(u) dW(u) - \frac{1}{2}\int_0^t \theta^2(u) du \Big]; \\ Z_2(t) &:= e^{(\lambda - \widetilde{\lambda})t} \left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N(t)}; \\ Z(t) &:= Z_1(t) Z_2(t). \end{aligned}$$

Also define

$$d\mathbb{Q} = Z(T) d\mathbb{P} \text{ on } \mathcal{F}(T).$$

Then $\widetilde{W}(t) = W(t) + \int_0^t \theta(u) du$ is a Brownian motion and $N(t)$ is a Poisson process with rate $\widetilde{\lambda}$ and they are independent under $\mathbb{Q}$.

6.2 Compound Poisson process and Brownian motion

Let $Q(t)$ be a compound Poisson process with rate λ and $W(t)$ a Brownian motion defined on the *same* probabilty space $(\mathbb{P}, \Omega, \mathcal{F})$ and $\mathcal{F}(t)$ a filtration for $Q(t), W(t)$. Here we also assume Y_i has continuous distribution with density function f.

We want to change the intensity of $Q(t)$, the distribution of Y_i (that is the density f) and the drift of $W(t)$ via the change of measure. More speficially, given a function $\theta(u)$, constant $\widetilde{\lambda} > 0$ and density function $\widetilde{f}$, we find the probabilty measure $\mathbb{Q}$ such that under $\mathbb{Q}$, $Q(t)$ is compound Poisson with rate $\widetilde{\lambda}$, $Y(i)$ has density $\widetilde{f}$ and $\widetilde{W}(t) := \int_0^t \theta(u)du + W(t)$ is a Brownian motion. Here we also assume that $\widetilde{f}(y) = 0$ when $f(y) = 0$.

Remark 6.3. *Before we proceed, we note that necessarily in this case $W(t)$ and $Q(t)$ are independent as remarked above (see also Corollary 11.4.9 and Exercise 11.6 in Shreve's).*

Definition 6.4. *Fix $T > 0$. Let $\widetilde{\lambda} > 0$ and a density function $\widetilde{f}$ be given. Define*

$$\begin{aligned} Z_1(t) &:= \exp\Big[-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du\Big]; \\ Z_2(t) &:= e^{(\lambda-\widetilde{\lambda})t}\prod_{i=1}^{N(t)}\left(\frac{\widetilde{\lambda}\widetilde{f}(Y_i)}{\lambda f(Y_i)}\right), 0 \le t \le T; \\ Z(t) &:= Z_1(t)Z_2(t). \end{aligned}$$

Also define

$$d\mathbb{Q} = Z(T)d\mathbb{P} \text{ on } \mathcal{F}(T).$$

Remark 6.5. *Note that $Z_1(t)$ is the usual change of measure kernel given by the Girsanov's theorem in Section 5.2. This together with the result in Section* (5.4) *and Remark* (6.3)*, it is no surprise that $Z(t)$ has such form.*

We have the important results: (Shreve's Lemma 11.6.8, Theorem 11.6.9)

Theorem 6.6. *Z(t) is a $\mathbb{P}$ martingale (w.r.t. $\mathcal{F}(t)$). Under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\widetilde{\lambda}$, Y_i has continuous distribution with density $\widetilde{f}$, $\widetilde{W(t)} = \int_0^t \theta(u)du + W(t)$ is a Brownian motion. Moreover, $Q(t)$ and $\widetilde{W(t)}$ are independent under $\mathbb{Q}$.*

Remark 6.7. *Note that we have the parallel between the independence between $Q(t)$ and $W(t)$ under $\mathbb{P}$ and the independence between $Q(t)$ and $\widetilde{W}(t)$ under $\mathbb{Q}$. This is important since we do not have any restriction on $\theta(t)$. Indeed $\theta(t)$ can be equal to $Q(t)$ and the independence structure still holds.*

Remark 6.8. *Even though the theorem in Shreve is stated for Y_i having continuous distribution, it is easy to see that a similar result still holds if Y_i has discrete distribution. In this case, under $\mathbb{Q}$, Y_i would also have discrete distribution with a probability distribution $\widetilde{p}$ (see Section* (5)*). The change of measure kernel $Z_1(t)$ is the same,*

$$Z_2(t) \quad := \quad e^{(\lambda-\widetilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\widetilde{\lambda}\widetilde{p}(Y_i)}{\lambda p(Y_i)}, 0 \leq t \leq T;$$

and $Z(t) = Z_1(t)Z_2(t)$.