Forward LIBOR model

Math 622

May 12, 2015

1 Forward LIBOR

1.1 Continuous vs simple compounding

Let $0 < t < S < T$. Suppose we want to price a product based on the forward rate, say a cap K on the forward rate for a loan taken on the interval from S to T, with the rate locked in at time t. The risk neutral pricing formula for this product would be

$$\tilde{E}\left(e^{-\int_S^T (f(t,u)-K)^+ du}\right).$$

(The meaning of such cap would be clearer when we discuss the Caplet below).

It is certainly desirable to be able to obtain a closed form solution to such formula, under the assumption that the volatility of $f(t,T)$ (as a process in t) is of the form $\sigma f(t,T)$ where σ is a constant. But we saw that from a purely mathematical point of view, this is not possible since it would force *the drift term* of $f(t,T)$ under the risk neutral measure $\tilde{P}$ to have certain form, which in turn makes the solution to $f(t,T)$ explode near T (Shreve's Section 10.4.1).

Furthermore, the problem of the drift term of $f(t,T)$ cannot be solved by a change of measure associated with a change of numéraire. You should try to derive the dynamics of $f(t,T)$ under the $\tilde{P}^{T'}$-forward measure, for some $T < T'$ for example, and convince yourself that the drift term of $f(t,T)$ cannot be eliminated.

The presence of the drift term, or equivalently the dynamics of $f(t,T)$, can be seen as coming from the *continuous compouding* used in its definition:

$$B(t,T) = e^{-\int_t^T f(t,u)du}.$$

It turns out that in order to "eliminate the drift term" in the "interest rate", so that we will be able to posit a log-normal distribution for it, we want to use *simple compounding* instead. That is, we denote $L_\delta(t,T)$ as the quantity that satisfies

$$B(t,T+\delta)(1+\delta L_\delta(t,T)) = B(t,T).$$

You should see that $L_\delta(t,T)$ is the interest rate one can lock in at time t for investing on the time interval $[T, T+\delta]$ with *simple compounding: repayment = investment × (1 + duration of investment × interest rate).*

$L_\delta(t,T)$ is called the simple forward LIBOR rate of tenor δ.

The reason why $L_\delta(t,T)$ should have a "better" dynamics than $f(t,T)$ (in terms of being able to posit a log normal distribution) is because of its definition:

$$1+\delta L_\delta(t,T) = \frac{B(t,T)}{B(t,T+\delta)}.$$

Thus clearly the dynamics of $L_\delta(t,T)$ is related to the dynamics of $B(t,T)$ under the $\tilde{P}^{T+\delta}$ forward measure, using $B(t,T+\delta)$ as numéraire. Since $D(t)B(t,T)$ is a martingale under $\tilde{P}$ we expect the dynamics of $L_\delta(t,T)$ under $\tilde{P}^{T+\delta}$ is "nice" as well.

Thus in this note we will develop the dynamics of $L_\delta(t,T)$ under the $T+\delta$ forward measure and show how to price financial products based on it (cap and caplet), under *the assumption of determinsitic volatility*, using Black-Scholes type of calculation.

1.2 How to construct a portfolio that realize the simple interest rate $L_\delta(t,T)$

Suppose at time $t<T$, we go short one share of $B(t,T)$ and long $B(t,T)/B(t,T+\delta)$ shares of $B(t,T+\delta)$. The value of this portfolio is zero at time t; at time T it requires us to pay out one dollar and at time $T+\delta$ we receive $B(t,T)/B(t,T+\delta)$ dollars. Thus at time t we can lock in a deposit that multiplies to $B(t,T)/B(t,t+\delta)$ over $[T,T+\delta]$ and hence earns the simple interest rate $L_\delta(t,T)$ satisfying

$$1+\delta L_\delta(t,T) = \frac{B(t,T)}{B(t,T+\delta)}$$

Thus

$$L_\delta(t,T) = \frac{1}{\delta}\left[\frac{B(t,T)}{B(t,T+\delta)} - 1\right] = \frac{1}{\delta}\frac{B(t,T)-B(t,T+\delta)}{B(t,T+\delta)}.$$

We have immediately that

$$1+\delta L_\delta(T,T)=\frac{1}{B(T,T+\delta)}.$$

Thus $L_\delta(T,T)$ is the simple interest rate available at time T for a deposit over time period $[T,T+\delta]$. This is a financially important quantity, because it is often used for floating rate loans or as a benchmark for interest rate caps and floors.

1.3 Dynamics of $L_\delta(t,T)$

Here is an elementary, but very important observation:

$$\begin{aligned} L_\delta(t,T) &= \frac{1}{\delta}\frac{B(t,T)-B(t,T+\delta)}{B(t,T+\delta)} \\ &= \frac{\frac{1}{\delta}B(t,T)-\frac{1}{\delta}B(t,T+\delta)}{B(t,T+\delta)}. \end{aligned}$$

Thus $L_\delta(t,T)$, for $t\le T$ is the $T+\delta$ forward price of a portfolio that is long $1/\delta$ zero coupon bonds that mature at T and short $1/\delta$ zero coupon bonds that mature at $T+\delta$.

In this section, we will derive the model implied for the forward LIBOR rate by the risk-neutral HJM model. To start out, observe that since

$$\begin{aligned} L_\delta(t,T) &= \frac{1}{\delta}\frac{B(t,T)-B(t,T+\delta)}{B(t,T+\delta)} \\ &= \frac{1}{\delta}\frac{B(t,T)}{B(t,T+\delta)}-\frac{1}{\delta}, \end{aligned}$$

we have

$$dL_\delta(t,T)=\delta^{-1}d[B(t,T)/B(t,T+\delta)].$$

Following the notation of the change of numéraire section, we define

$$B^{T+\delta}(t,T):=B(t,T)/B(t,T+\delta)$$

as the $T+\delta$ forward price of $B(t,T)$.

Observe then, that it is most natural to express the model for $L_\delta(t,T)$ under the $T+\delta$ forward measure $\widetilde{\mathbf{P}}^{T+\delta}$. We know from Theorems 9.2.1 and 9.2.2 in Shreve that because

$$\begin{aligned} dD(t)B(t,T) &= -D(t)B(t,T)\sigma^*(t,T)\,d\widetilde{W}(t) \\ dD(t)B(t,T+\delta) &= -D(t)B(t,T+\delta)\sigma^*(t,T+\delta)\,d\widetilde{W}(t), \end{aligned}$$

we have

$$\begin{aligned} dL_\delta(t,T) &= \frac{1}{\delta} B^{T+\delta}(t,T)[\sigma^*(t,T+\delta) - \sigma^*(t,T)]\, d\widetilde{W}^{T+\delta}(t) \\ &= \frac{1}{\delta}\,[1+\delta L_\delta(t,T)]\,[\sigma^*(t,T+\delta) - \sigma^*(t,T)]\, d\widetilde{W}^{T+\delta}(t) \\ &= L_\delta(t,T)\left\{\frac{1+\delta L_\delta(t,T)}{\delta L_\delta(t,T)}[\sigma^*(t,T+\delta) - \sigma^*(t,T)]\right\} d\widetilde{W}^{T+\delta}(t), \end{aligned} \tag{1}$$

where $\widetilde{W}^{T+\delta}(t) = \widetilde{W}(t) + \int_0^t \sigma^*(u, T+\delta)\, du$ is a Brownian motion under $\widetilde{\mathbf{P}}^{T+\delta}$. From this equation we can easily derive the model for the forward LIBOR rate under the original risk-neutral measure $\widetilde{\mathbf{P}}$, but we will not have need for this.

Remark:
(i) If we denote

$$\gamma(t) := \frac{1+\delta L_\delta(t,T)}{\delta L_\delta(t,T)}[\sigma^*(t,T+\delta) - \sigma^*(t,T)],$$

then it follows that

$$dL_\delta(t,T) = L_\delta(t,T)\gamma(t) d\widetilde{W}^{T+\delta}(t).$$

If we **assume** $\gamma(t)$ is a constant then it is easy to see that $L_\delta(t,T)$ has log-normal distribution under $\widetilde{\mathbf{P}}^{T+\delta}$, which is a goal we have set out to achieve. This will help us to derive pricing equation in Black-Scholes style for financial products based on $L_\delta(t,T)$ as discussed in the Sections below.

(ii) Assuming $\gamma(t)$ is a constant is a **big** assumption if we start from the risk neutral model of $B(t,T)$ and $B(t,T+\delta)$. However, we can start modeling under the $T+\delta$-forward measure, where we are *free to assume* the fact that $\gamma(t)$ is a constant. The distribution of $B(t,T)$ and $B(t,T+\delta)$ under the risk neutral measure can then be derived from the $\tilde{P}^{T+\delta}$ model.

2 T-forward models

Previously, we defined a T-forward measure. This is a measure, $\widetilde{\mathbf{P}}^T$, if it exists, under which T-forward prices of all market assets are martingales. Recall that the T-forward price of an asset whose price in dollars is $S(t)$ is $S(t)/B(t,T)$. Now assume we have an HJM model driven by a single Brownian motion, and write it under the risk-neutral measure $\widetilde{\mathbf{P}}$. According to the theory developed in Chapter 9 of Shreve, the T-forward measure is defined by a change of measure from $\widetilde{\mathbf{P}}$ by the Radon-Nikodym derivative,

$$\frac{d\widetilde{\mathbf{P}}^T}{d\widetilde{\mathbf{P}}} = \frac{D(T)}{B(0,T)}. \tag{2}$$

That is, $\widetilde{\mathbf{P}}^T(A) = E[\mathbf{1}_A D(T)]/B(0,T)$, for $A \in \mathcal{F}$. But we know the solution to

$$dB(t,T) = R(t)B(t,T)dt - \sigma^*(t,T)B(t,T)dWt$$

is

$$D(t)B(t,T) = B(0,T)\exp\{-\int_0^t \sigma^*(u,T)\,dW(u) - \frac{1}{2}\int_0^t (\sigma^*)^2(u,T)\,du\}$$

and hence

$$\frac{d\widetilde{\mathbf{P}}^T}{d\widetilde{\mathbf{P}}} = \exp\{-\int_0^T \sigma^*(u,T)\,dW(u) - \frac{1}{2}\int_0^T (\sigma^*)^2(u,T)\,du\}. \tag{3}$$

It follows from Girsanov's theorem that

$$\widetilde{W}^T(t) = \widetilde{W}(t) + \int_0^t \sigma^*(u,T)\,du \tag{4}$$

is a Brownian motion under $\widetilde{\mathbf{P}}^T$, at least for times $t \leq T$.

All this is review of section 9.4 in Shreve.

3 Financial products based on forward LIBOR

3.1 Description

The forward LIBOR $L_\delta(t,T)$ is strictly not a financial asset by itself. However, if we think about investing a principal P at time T for the duration $[T, T+\delta]$ to realize the interest payment $P\delta L_\delta(T,T)$ at time $T+\delta$, then we have a product that is very much like a Euro style derivative, with expiry $T+\delta$.

One can also create another product that is in the spirit of the Euro Call option, in this case called an *interest rate cap*. For a constant K positive, we can consider a financial product that pays

$$V_{T+\delta} = \delta P\big(L_\delta(T,T) - K\big)^+$$

at time $T+\delta$. The interpretation is that if we borrow an amount P at time T, we may not want the interest rate $L_\delta(T,T)$ to go beyond K. Therefore to protect ourselves, we would want to get an interest rate cap that would pay us the difference should the interest rate go beyond K.

Moreover, since P and δ are deterministic (we think of them as determined at time 0), for simplicity we can take $P\delta = 1$. Thus, one can discuss the following products:

(i) A contract that pays $L_\delta(T,T)$ at time $T+\delta$. This is called a backset LIBOR on a notional amount of 1.

(ii) A contract that pays $(L_\delta(T,T)-K)^+$ at time $T+\delta$. This is called an *interest rate caplet.*

Clearly the question is what are the risk neutral prices of these products at time 0. We will give the formula for backset LIBOR in this section and give a detailed discussion of interest rate cap and caplet in the next section.

3.2 Risk neutral price of backset LIBOR

Theorem 3.1. *The no arbitrage price at time t of a contract that pays $L_\delta(T,T)$ at time $T+\delta$ is*

$$\begin{aligned} S(t) &= B(t,T+\delta)L_\delta(t,T),\ 0 \le t \le T \\ &= B(t,T+\delta)L_\delta(T,T),\ T \le t \le T+\delta. \end{aligned}$$

($S(t)$ is the notation Shreve used in the textbook. Don't confuse it with the stock price).

Proof:

By the risk neutral pricing formula

$$S(t) = \widetilde{E}\Big[e^{-\int_t^{T+\delta} R(u)du} L_\delta(T,T)\Big|\mathcal{F}(t)\Big].$$

If $T \le t$ then $L_\delta(T,T)$ is $\mathcal{F}(t)$ measurable. Therefore

$$S(t) = L_\delta(T,T)\widetilde{E}\Big[e^{-\int_t^{T+\delta} R(u)du}\Big|\mathcal{F}(t)\Big] = B(t,T+\delta)L_\delta(T,T).$$

If $t < T$ then by the change of numéraire pricing formula under $\widetilde{\mathbf{P}}^{T+\delta}$ we have

$$\frac{S(t)}{B(t,T+\delta)} = \widetilde{E}^{T+\delta}\Big[L_\delta(T,T)\Big|\mathcal{F}(t)\Big].$$

But $L_\delta(t,T)$ is a martingale under $\widetilde{\mathbf{P}}^{T+\delta}$ (see equation 1 in Section 1). Therefore,

$$\frac{S(t)}{B(t,T+\delta)} = L_\delta(t,T)$$

and the conclusion follows.

4 Caps and caplets

4.1 Description

We will consider the following type of floating rate bond. It starts at $T_0 = 0$ and pays coupons $C_1,\dots,C_{n+1}$ on principal P at dates $T_1 = \delta$, $T_2 = 2\delta,\dots,T_j = j\delta,\dots,T_{n+1} = (n+1)\delta$. The interest charged over $[T_{j-1},T_j]$ is the LIBOR rate set at T_{j-1}. So coupon $C_j = \delta P L_\delta(T_{j-1},T_{j-1})$.

Suppose now that Alice has issued such a bond. An equivalent interpretation is *she has taken out a floating rate loan.* For convenience, assume the principal is \$1. She can purchase an *interest rate cap* to protect herself against unacceptable increases in the floating rate.

A cap set at strike K and lasting until T_{n+1} will pay her $\delta(L_\delta(T_{j-1},T_{j-1})-K)^+$ at each time T_j, $1 \le j \le n+1$. This means that she will never pay more than rate K over any period; the cap will make up the difference between the $\delta L_\delta(T_{j-1},T_{j-1})$ she owes the bond holder and the maximum δK she wishes to pay. We shall use $\mathrm{Cap}^{\mathrm{m}}(0,n+1)$ to denote the market price of this cap at time $T_0 = 0$.

Consider the derivative which pays the interest rate cap only at time T_j. So it consists of *a single payoff* $\delta(L_\delta(T_{j-1},T_{j-1})-K)^+$ at T_j. This is called a *caplet.* Caplets are not traded as such. However, we can imagine them for the purposes of pricing. Clearly, if $\mathrm{Caplet}_j(0)$ denotes the price of this caplet at time $T_0 = 0$, the total price at $T_0 = 0$ of a cap of maturity T_{n+1} will be

$$\sum_{j=1}^{n+1} \mathrm{Caplet}_j(0).$$

If caps of all maturities are available on the market, we can create a caplet with payoff at T_j by going long one cap maturing at T_j and short one cap maturing at T_{j-1}. Thus the market price of the caplet at T_j is

$$\mathrm{Caplet}_j(0) = \mathrm{Cap}^m(0,j) - \mathrm{Cap}^m(0,j-1).$$

Just as there are interest rate caps, there are also interest floors. By going long a cap and short a floor, one can create also a *collar* that keeps the interest rate one pays between two levels.

Interest rate caps and floors are widely traded and their prices are readily available from the market.

4.2 A remark on the Black-Scholes formula

The pricing formula for the caplet follows the argument of the Black-Scholes formula. The derivation of the Black-Scholes formula is a direct consequence of the following result about normal random variables, which in turn is a consequence of Corollary 1 in the class lecture notes, *Review of Mathematical Finance I.*

Theorem 1. *If Y is a normal random variable with mean 0 and variance ν^2,*

$$E\left[\left(xe^{Y-\nu^2/2}-K\right)^+\right]=xN\left(\frac{\ln(x/K)+\nu^2/2}{\nu}\right)-KN\left(\frac{\ln(x/K)-\nu^2/2}{\nu}\right). \quad (5)$$

To see the connection to the Black-Scholes formula, note that the price at time 0 of a call with strike K is

$$e^{-rT}\tilde{E}\left[\left(xe^{\sigma\widetilde{W}(T)+rT-\frac{1}{2}\sigma^2T}-K\right)^+\right]=e^{-rT}\tilde{E}\left[\left(xe^{rT}e^{\sigma\widetilde{W}(T)-\frac{1}{2}\sigma^2T}-K\right)^+\right].$$

Since $\sigma\widetilde{W}(T)$ is a normal random variable with mean 0 and variance σ^2T, we are exactly in the situation of Theorem 1, and it is easy to derive the Black-Scholes formula from (5).

4.3 Black's caplet model and pricing formula

The idea behind Black's caplet model and price is to take advantage of Theorem 1 by positing lognormal models where possible. We already saw this strategy in section 9.4 of Shreve, where we assumed T-forward prices for a given T were lognormal. The idea for caplets is similar. Consider the caplet that pays $\delta(L_\delta(T_j,T_j)-K)^+$ at T_{j+1}. We posit that there is a risk-neutral model $\widetilde{\mathbf{P}}^{T_{j+1}}$ under which T_{j+1} forward prices are martingales, that there is a Brownian motion $\widetilde{W}^{T_{j+1}}$ under $\widetilde{\mathbf{P}}^{T_{j+1}}$ and that

$$dL_\delta(t,T_j)=\gamma(t,T_j)L_\delta(t,T_j)\,d\widetilde{W}^{T_{j+1}}, \quad (6)$$

where $\gamma(t, T_j)$ is deterministic. Equivalently,

$$L_\delta(t, T_j) = L_\delta(0, T_j) \exp\left\{ \int_0^t \gamma(u, T_j)\, d\widetilde{W}^{T_{j+1}}(u) - \frac{1}{2} \int_0^t \gamma^2(u, T_j)\, du \right\}.$$

For convenience of notation, let

$$\bar{\gamma}^2(T_j) = \frac{1}{T_j} \int_0^{T_j} \gamma^2(u, T_j)\, du.$$

Let $\mathbf{Caplet}_{j+1}(0, \bar{\gamma}(T_j))$ denote the price at $T_0 = 0$ of the caplet maturing at T_{j+1}); (we will see that this price depends only on $\bar{\gamma}(T_j)$, if δ and K are fixed, so the notation is appropriate.) By the risk-neutral pricing formula, the T_{j+1}-forward price of the caplet is

$$\frac{\mathbf{Caplet}_{j+1}(0, \bar{\gamma}(T_j))}{B(0, T_{j+1})} = \delta \tilde{E}^{T_{j+1}} \left[\left(L_\delta(0, T_j) e^{\int_0^{T_j} \gamma(u,T_j)\, d\widetilde{W}^{T_{j+1}}(u) - \frac{1}{2}\int_0^{T_j} \gamma^2(u,T_j)\, du} - K \right)^+ \right].$$

But, since $\gamma(t, T_j)$ is deterministic, $\int_0^{T_j} \gamma(u, T_j)\, d\widetilde{W}^{T_{j+1}}(u)$ is a normal random variable with mean 0 and variance $\int_0^{T_j} \gamma^2(u, T_j)\, du = T_j \bar{\gamma}(T_j)$. Thus from Theorem 1,

$$\begin{aligned} \frac{\mathbf{Caplet}_{j+1}(0, \bar{\gamma}(T_j))}{B(0, T_{j+1})} &= \delta L_\delta(0, T_j) N\left(\frac{\ln \frac{L_\delta(0,T_j)}{K} + \frac{1}{2}\bar{\gamma}^2(T_j) T_j}{\bar{\gamma}(T_j)\sqrt{T_j}} \right) \\ &\quad - \delta K N\left(\frac{\ln \frac{L_\delta(0,T_j)}{K} - \frac{1}{2}\bar{\gamma}^2(T_j) T_j}{\bar{\gamma}(T_j)\sqrt{T_j}} \right) \end{aligned}$$

In this way, we derive *Black's caplet formula*:

$$\begin{aligned} \mathbf{Caplet}_{j+1}(0, \bar{\gamma}(T_j)) &= B(0, T_{j+1}) \Bigg[\delta L_\delta(0, T_j) N\left(\frac{\ln \frac{L_\delta(0,T_j)}{K} + \frac{1}{2}\bar{\gamma}^2(T_j) T_j}{\bar{\gamma}(T_j)\sqrt{T_j}} \right) \\ &\quad - \delta K N\left(\frac{\ln \frac{L_\delta(0,T_j)}{K} - \frac{1}{2}\bar{\gamma}^2(T_j) T_j}{\bar{\gamma}(T_j)\sqrt{T_j}} \right) \Bigg] \end{aligned} \tag{7}$$

The implied spot volatility is a number γ_j, which, when substituted into Black's caplet formula, gives the market value:

$$\mathbf{Caplet}_{j+1}(0, \gamma_j) = \text{Caplet}_{j+1}(0).$$

By finding the implied volatilities and then choosing $\gamma(t, T_j)$ for each j so that

$$\int_0^{T_j} \gamma^2(u, T_j)\, du = T_j \gamma_j^2,$$

we can fit Black's model to the market for all j.

We emphasize that this model is formulated directly for forward LIBOR and does not assume that one has formulated a prior model, such as an HJM model, for zero-coupon bond prices.

5 Calibration of forward LIBOR model

5.1 Motivation

From the above, we've seen that the forward LIBOR rates for different maturity $T_j, 1 \leq j \leq n$ have the dynamics:

$$dL_\delta(t, T_j) = \gamma(t, T_j) L_\delta(t, T_j)\, d\widetilde{W}^{T_{j+1}},$$

where $\widetilde{W}^{T_{j+1}}$ is a Brownian motion under the T_{j+1} forward measure.

The financial products associated with these LIBOR rates are the caplets that pay $\delta(L_\delta(T_{j-1}, T_{j-1}) - K)^+$ at T_j. The market price of these caplets can be derived from the price of the caps:

$$\text{Caplet}_j(0) = \text{Cap}^m(0, j) - \text{Cap}^m(0, j-1).$$

On the other hand, from the model of the LIBOR rates, we can also derive, under the assumption that $\gamma(t, T_j)$ are determinstic, via Black-Scholes formula, the theoretical price of these caplets. We denote these prices by $\mathbf{Caplet}_j(0, \bar{\gamma}(T_{j-1}))$.

The obvious question is: can we build a model of these forward LIBOR rates so that

$$\text{Caplet}_j(0) = \mathbf{Caplet}_j(0, \bar{\gamma}(T_{j-1}))?$$

The answer is of course yes. Since $\mathbf{Caplet}_j(0, \bar{\gamma}(T_{j-1}))$ is a function of $\bar{\gamma}(T_{j-1}))$ we can choose a number γ_{j-1} so that the above equation holds:

$$\text{Caplet}_j(0) = \mathbf{Caplet}_j(0, \gamma_{j-1}).$$

γ_{j-1} is called the implied volatility of the LIBOR rate with maturity T_j. In general, we do not have an explicit formula for γ_{j-1}. The way to find γ_{j-1} is via numerical procedure, but it can be done.

Next we can construct a determinsitic function $\gamma(t, T_{j-1})$ so that

$$\int_0^{T_{j-1}} \gamma^2(t, T_{j-1}) = T_{j-1}\gamma_{j-1}.$$

There is much freedom in choosing $\gamma(t, T_{j-1})$ of course.

We may think the next step is just to construct n Brownian motions: $\widetilde{W}^{T_{j+1}}, 1 \leq j \leq n$ (question: how are they related?) and from which we can derive n LiBOR rates $L_\delta(t, T_j), 1 \leq j \leq n$ from which the Caplet price will match the market data. But this is missing some details.

First, we want to build a consistent model for $L_\delta(t, T_j), 1 \leq j \leq n$, beyond just matching the market data at time 0. Recall the definition of the LIBOR rates:

$$\begin{aligned} L_\delta(t, T_j) &= \frac{1}{\delta}\frac{B(t, T_j) - B(t, T_j + \delta)}{B(t, T_j + \delta)} \\ &= \frac{1}{\delta}\frac{B(t, T_j) - B(t, T_{j+1})}{B(t, T_{j+1})} \end{aligned}$$

So even without knowing the details, we should suspect that $L_\delta(t, T_j)$ and $L_\delta(t, T_{j+1})$ are related at some level. If we simulate $\widetilde{W}^{T_{j+1}}$ and $\widetilde{W}^{T_{j+2}}$ without regards to this relation, we're missing certain things.

Second, suppose starting out from the risk neutral measure $\widetilde{P}$, we have the dynamics of the bond $B(t, T_j)$ as

$$dB(t, T_j) = R(t)B(t, T_j)dt + \sigma^*(t, T_j)B(t, T_j)\widetilde{W}(t).$$

Note that there is only one Brownian motion $\widetilde{W}$ here, which is *independent* of T_j. (The choice of how many Brownian motions we put in is up to us, of course, but the point is that we use the same Brownian motions to model the dynamics of $B(t, T_j)$ for different T_j). So from what we learned from the change of numéraire section, the Brownian motion $\widetilde{W}^{T_{j+1}}$ are all related to $\widetilde{W}$ via the equation:

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}(t) + \sigma^*(t, T_j)\, dt.$$

Thus all Brownian motions $\widetilde{W}^{T_j}$ are related actually. So to model $L_\delta(t, T_j)$ properly, beyond determining the $\gamma(t, T_j)$ to match the market data, we also need to learn about the relations of $L_\delta(t, T_j)$. We will do so in the next section.

Finally, as the bond price $B(t, T_j)$ and LIBOR rates $L_\delta(t, T_j)$ are clearly related, we will see that by modeling the $L_\delta(t, T_j)$ properly, this will also give us a handle on how to model the volatility $\sigma^*(t, T_j)$ of the bonds and the (discounted) value of the bond $B(t, T_j)$ themselves. The details will be given in the third section.

5.2 Consistent forward LIBOR models - Relation among the $L(t, T_j)$

5.2.1 Relation among $\widetilde{W}^{T_j}$

Recall from section (5.1) that for every j, $d\widetilde{W}^{T_j}(t) = d\widetilde{W}(t) + \sigma^*(t, T_j)\, dt$. In particular, it follows from this that

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}^{T_{j+1}}(t) + [\sigma^*(t, T_j) - \sigma^*(t, T_{j+1})]\, dt \tag{8}$$

Next, from the dynamics of $L_\delta(t, T_j)$ that we derived before:

$$dL_\delta(t, T_j) = L_\delta(t, T_j) \left\{ \frac{1 + \delta L_\delta(t, T_j)}{\delta L_\delta(t, T_j)} [\sigma^*(t, T_{j+1}) - \sigma^*(t, T_j)] \right\} d\widetilde{W}^{T_{j+1}}(t).$$

This will be the same as the Black model $dL_\delta(t, T_j) = \gamma(t, T_j) L_\delta(t, T_j)\, d\widetilde{W}^{T_{j+1}}(t)$ only if

$$\gamma(t, T_j) = \frac{1 + \delta L_\delta(t, T_j)}{\delta L_\delta(t, T_j)} [\sigma^*(t, T_{j+1}) - \sigma^*(t, T_j)], \qquad t \leq T_j$$

or equivalently,

$$\sigma^*(t, T_{j+1}) - \sigma^*(t, T_j) = \frac{\delta L_\delta(t, T_j)}{1 + \delta L_\delta(t, T_j)} \gamma(t, T_j), \quad t \leq T_j. \tag{9}$$

Assume this is the case for all $j \leq n$. By combining this result with equation (8),

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}^{T_{j+1}}(t) - \frac{\delta L_\delta(t, T_j)}{1 + \delta L_\delta(t, T_j)} \gamma(t, T_j)\, dt \tag{10}$$

The significance of this equation is that *the processes $\sigma^*(t, T)$ no longer explicitly appear—everything is expressed in terms of the LIBOR rates themselves and their volatility functions $\gamma(t, T_j)$.*

By working backward with (10), $d\widetilde{W}^{T_j}(t)$ can be expressed in terms of $d\widetilde{W}^{T_{n+1}}(t)$ for all j. Indeed,

$$d\widetilde{W}^{T_n}(t) = d\widetilde{W}^{T_{n+1}}(t) - \frac{\delta L_\delta(t, T_n)}{1 + \delta L_\delta(t, T_n)} \gamma(t, T_n)\, dt.$$

But then

$$\begin{aligned} d\widetilde{W}^{T_{n-1}}(t) &= d\widetilde{W}^{T_n}(t) - \frac{\delta L_\delta(t, T_{n-1})}{1 + \delta L_\delta(t, T_{n-1})} \gamma(t, T_{n-1})\, dt \\ &= d\widetilde{W}^{T_{n+1}}(t) - \left[\frac{\delta L_\delta(t, T_n)}{1 + \delta L_\delta(t, T_n)} \gamma(t, T_n) + \frac{\delta L_\delta(t, T_{n-1})}{1 + \delta L_\delta(t, T_{n-1})} \gamma(t, T_{n-1}) \right] dt. \end{aligned}$$

Continuing further, and using what has just been derived,

$$\begin{aligned} d\widetilde{W}^{T_{n-2}}(t) &= d\widetilde{W}^{T_{n-1}}(t) - \frac{\delta L_\delta(t,T_{n-2})}{1+\delta L_\delta(t,T_{n-2})}\gamma(t,T_{n-2})\,dt \\ &= d\widetilde{W}^{T_{n+1}}(t) - \Bigg[\sum_{i=n-2}^{n} \frac{\delta L_\delta(t,T_i)}{1+\delta L_\delta(t,T_i)}\gamma(t,T_i)\Bigg]\,dt. \end{aligned}$$

Clearly, this will yield for general $j \le n$ that

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}^{T_{n+1}}(t) - \Bigg[\sum_{i=j}^{n} \frac{\delta L_\delta(t,T_i)}{1+\delta L_\delta(t,T_i)}\gamma(t,T_i)\Bigg]\,dt. \tag{11}$$

The significance of this equation, compare with (10) is that *now all* $\widetilde{W^{T_j}}$ *is written in terms of* $\widetilde{W^{T_{n+1}}}$. *Thus, instead of generating n Brownian motions, we only need to generate one Brownian motion* $\widetilde{W^{T_{n+1}}}$. *This is consistent with what we mentioned before that we started out with only one Brownian Motion under risk neutral measure* $\widetilde{W}$.

5.2.2 The relation among the $L(t,T_j)$ - Their construction

Now we can write down a coherent system of equations for the LIBOR forward rates . First of all, Black's model for $j = n$ gives

$$dL_\delta(t,T_n) = L_\delta(t,T_n)\gamma(t,T_n)\,d\widetilde{W}^{T_{n+1}}(t), \qquad t \le T_n. \tag{12}$$

Next, for arbitrary $j < n$, $dL_\delta(t,T_j) = L_\delta(t,T_j)\gamma(t,T_j)\,d\widetilde{W}^{T_{j+1}}(t)$, and so

$$dL_\delta(t,T_j) = L_\delta(t,T_j)\gamma(t,T_j)\Bigg[-\sum_{i=j+1}^{n} \frac{\delta L_\delta(t,T_i)}{1+\delta L_\delta(t,T_i)}\gamma(t,T_i) + d\widetilde{W}^{T_{n+1}}(t)\Bigg], \qquad t \le T_j \tag{13}$$

This system of equations makes no reference to the original risk-neutral HJM model. In fact, it can stand alone as its own model. By working backwards on this set of equations using standard theorems, one can prove that it generates a consistent model for caplets of all maturities up to T_{n+1}, without assuming the prior existence of an HJM model for $B(t,T)$. We state this result and summarize the forward LIBOR model in the following theorem. The proof will be given in the next subsection where we discuss the relation between forward measures, see (3).

Theorem 2. *Let there be given a probability space with measure* $\widetilde{\mathbf{P}}^{T_{n+1}}$ *supporting a Brownian motion* $\widetilde{W}^{T_{n+1}}$. *Then there exists a unique solution* $L_\delta(t,T_1),\ldots,L_\delta(t,T_n)$

to the system of equations (12)–(13). If the measures $\widetilde{\mathbf{P}}^{T_j}$*,* $j = n, n-1, \ldots, 1$ *are defined recursively by*

$$\widetilde{\mathbf{P}}^{T_j}(A) = \tilde{E}^{T_{j+1}}\left[\mathbf{1}_A \frac{1+\delta L_\delta(T_j, T_j)}{1+\delta L_\delta(0, T_j)}\right],$$

and the processes $\widetilde{W}^{T_j}(t)$*,* $1 \le j \le n$*, are defined recursively by*

$$d\widetilde{W}^{T_j}(t) = d\widetilde{W}^{T_{j+1}}(t) - \frac{\delta L_\delta(t, T_j)}{1+\delta L_\delta(t, T_j)}\gamma(t, T_j)\, dt,$$

then $\widetilde{W}^{T_j}$ *is a Brownian motion under* $\widetilde{\mathbf{P}}^{T_j}$ *for each* $j \le n$ *and*

$$dL_\delta(t, T_j) = L_\delta(t, T_j)\gamma(t, T_j)\, d\widetilde{W}^{T_{j+1}}(t), \quad \textit{for each } j \le n.$$

5.2.3 Changing between T-forward measures

Let $0 < T < T'$. Suppose that we have a risk-neutral model for the T' forward prices of a market in which zero-coupon bonds are offered on all maturities. We are not assuming that this has necessarily been derived from an HJM model, just that we have a probability space with a measure $\widetilde{\mathbf{P}}^{T'}$ under which the T'-forward prices of all assets are martingales. Let us denote the T' forward price of an asset whose price in dollars is $S(t)$ by $S^{T'}(t) = S(t)/B(t, T')$. In particular, the T'-forward price of a zero-coupon bond maturing at T, which is

$$B^{T'}(t, T) = \frac{B(t, T)}{B(t, T')}, \quad t \le T,$$

is a martingale under $\widetilde{\mathbf{P}}^{T'}$. The T forward price of an asset whose T' forward price is $S^{T'}(t)$ is

$$S^T(t) = \frac{S(t)}{B(t, T)} = \frac{S(t)/B(t, T')}{B(t, T)/B(t, T')} = \frac{S^{T'}(t)}{B^{T'}(t, T)}.$$

We are interested in finding the $\widetilde{\mathbf{P}}^T$-forward measure that makes prices $S^T(t)$ into martingales. Since we are not starting from an HJM model as in the previous section, we want to derive this in terms of the T'-forward measure. Denote expectation with respect to $\widetilde{\mathbf{P}}^{T'}$ by $\tilde{E}^{T'}$.

Theorem 3. *Define,* $\widetilde{\mathbf{P}}^T$ *by*

$$\widetilde{\mathbf{P}}^T(A) = \frac{B(0, T')}{B(0, T)}\tilde{E}^{T'}[\mathbf{1}_A \frac{1}{B(T, T')}] \tag{14}$$

Then if an asset is such that its T'- forward price is a martingale under $\widetilde{\mathbf{P}}^{T'}$ then its T-forward price is also a martingale under $\widetilde{\mathbf{P}}^{T}$.

This theorem is a generalization of formula (9.2.7) in Shreve.

Heuristic idea:

The intuitive idea why formula (14) is true is as followed. We want to convert from $\widetilde{\mathbf{P}}^{T'}$ to $\widetilde{\mathbf{P}}^{T}$. The numéraire associated with $\widetilde{\mathbf{P}}^{T}$ is $B(t,T)$. The price process of this numéraire under $\widetilde{\mathbf{P}}^{T'}$ is

$$N(t) := \frac{B(t,T)}{B(t,T')}.$$

Thus the change of measure formula states that

$$\begin{aligned} \widetilde{\mathbf{P}}^{T}(A) &= \tilde{E}^{T'}[\mathbf{1}_A \frac{N(T)}{N(0)}] \\ &= \frac{B(0,T')}{B(0,T)} \tilde{E}^{T'}[\mathbf{1}_A \frac{1}{B(T,T')}]. \end{aligned}$$

Compare this with what we did for change of measure from $\widetilde{\mathbf{P}}$ to $\widetilde{\mathbf{P}}^{(N)}$, for example. The numéraire under $\widetilde{\mathbf{P}}^{(N)}$ is clearly $N(t)$. Its "price" under $\widetilde{\mathbf{P}}$ is $D(t)N(t)$. Therefore the change of measure formula is

$$\widetilde{\mathbf{P}}^{(N)}(A) = \widetilde{E}[\mathbf{1}_A \frac{D(T)N(T)}{D(0)N(0)}]$$

Rigorous proof:

The proof is an application of Lemma 5.2.2 in Shreve: Suppose that $Z(t)$ is a positive martingale under a probability measure $\mathbf{P}$ and define

$$\mathbf{P}^Z(A) = E[\mathbf{1}_A Z(T)]/Z(0).$$

Then if $M(t)$ is a martingale under $\mathbf{P}$,

$$\{M(t)/Z(t);\, t \leq T\}$$

is a martingale under $\mathbf{P}^Z$. To prove the theorem, simply apply this principle with $\widetilde{\mathbf{P}}$ in place of $\mathbf{P}$ and $B^{T'}(t,T) = B(t,T)/B(t,T')$ in place of $Z(t)$. Note that the definition in (14) is the same as

$$\widetilde{\mathbf{P}}^T(A) = \tilde{E}^{T'}[\mathbf{1}_A B^{T'}(T,T)]/B^{T'}(0,T).$$

Since a T' forward price $S^{T'}(t)$ is a martingale under $\widetilde{\mathbf{P}}^{T'}$, it follows that the T forward price

$$S^T(t) = S^{T'}(t)/B^{T'}(t,T),$$

is a martingale under $\widetilde{\mathbf{P}}^T$ as defined in (14). This completes the proof.

5.3 Construction the T_j-Maturity Discounted Bonds

5.3.1 Construction of $\sigma^*(t,T_j)$

The above theorem does not give us a HJM model, which is defined in terms of functions $\sigma^*(t,T_j)$ on the risk-neutral probability for prices denominated in the domestic currency. This is done in Shreve on pages 444-447. We will only outline the main idea here.

With the deterministic functions $\gamma(t,T_j)$ in hand, we can construct the functions $\sigma^*(t,T_j)$ that are consistent with $\gamma(t,T_j)$

$$\sigma^*(t,T_{j+1}) - \sigma^*(t,T_j) = \frac{\delta L_\delta(t,T_j)}{1+\delta L_\delta(t,T_j)}\gamma(t,T_j) \quad t \le T_j.$$

By writing this as

$$\sigma^*(t,T_{j+1}) = \sigma^*(t,T_j) + \frac{\delta L_\delta(t,T_j)}{1+\delta L_\delta(t,T_j)}\gamma(t,T_j), \quad t \le T_j. \tag{15}$$

we see that $L_\delta(t,T_j)$, $\gamma(t,T_j)$, and $\sigma^*(t,T_j)$ *determine* $\sigma^*(t,T_{j+1})$ at least for $t \le T_j$. This leads to a recursive procedure for defining $\sigma^*(t,T_j)$. We outline the procedure of construction here:

1. Choose $\sigma^*(t,T_1)$ for $0 \le t \le T_1$. The only constraint is

$$\lim_{t\to T_1} \sigma^*(t,T_1) = 0.$$

2. Construct $\sigma^*(t,T_2)$ for $0 \le t \le T_1$ (note the time interval) using the relation

$$\sigma^*(t,T_{j+1}) = \sigma^*(t,T_j) + \frac{\delta L_\delta(t,T_j)}{1+\delta L_\delta(t,T_j)}\gamma(t,T_j), \quad t \le T_j.$$

3. Choose $\sigma^*(t, T_2)$ for $T_1 \leq t \leq T_2$ (again note the time interval). The only constraint is

$$\lim_{t \to T_2} \sigma^*(t, T_2) = 0.$$

4. Repeat this procedure to construct $\sigma^*(t, T_j)$ for $j \geq 3$.

Observe that in the above procedure, we had freedom to construct $\sigma^*(t, T_j)$ on the interval $T_{j-1} \leq t \leq T_j$ subject to the only constraint

$$\lim_{t \to T_j} \sigma^*(t, T_j) = 0.$$

Thus there is also much freedom in constructing $\sigma^*(t, T_j)$.

5.3.2 Construction the T_j-Maturity Discounted Bonds

Now that we have constructed $\sigma^*(t, T_j)$, the dynamics of the bond $B(t, T_j)$ under the risk neutral measure $\widetilde{P}$ is straightforward:

$$dB(t, T_j) = R(t)B(t, T_j)dt - \sigma^*(t, T_j)B(t, T_j)d\widetilde{W}(t).$$

Since we constructed the LIBOR rate under the forward measure $\widetilde{P}^{T_{n+1}}$ and the Brownian motion $\widetilde{W}^{T_{n+1}}$, it's also convenient to write the dynamics of $B(t, T_j)$ using these as well:

$$dB(t, T_j) = R(t)B(t, T_j)dt + \sigma^*(t, T_j)\sigma^*(t, T_{n+1})B(t, T_j)dt - \sigma^*(t, T_j)B(t, T_j)d\widetilde{W}^{T_{n+1}}(t).$$

Lastly, since we haven't constructed $R(t)$, it is better to write the dynamics of the discounted bond price instead:

$$d\Big(D(t)B(t, T_j)\Big) = \sigma^*(t, T_j)\sigma^*(t, T_{n+1})D(t)B(t, T_j)dt - \sigma^*(t, T_j)D(t)B(t, T_j)d\widetilde{W}^{T_{n+1}}(t).$$

We need the initial conditions to generate the bonds. They can be obtained from the LIBOR rates we have constructed as well:

$$D(0)B(0, T_j) = B(0, T_j) = \prod_{i=0}^{j-1} \frac{B(0, T_{i+1})}{B(0, T_i)} = \prod_{i=0}^{j-1} \Big(1 + \delta L_\delta(0, T_i)\Big)^{-1}.$$

The verification that our construction is consistent: $D(t)B(t, T_j)$ is a martingale under $\widetilde{P}$ is stated in Shreve's Theorem 10.4.4. (The only subtle point is we start out modeling under the forward measure $\widetilde{P}^{T_{n+1}}$. So we need to *define* the risk neutral measure $\widetilde{P}$ from $\widetilde{P}^{T_{n+1}}$. After that the verification is straightforward.)