Jump process models

Math 622 - Spring 2015

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1 Motivations

Previously, most of the models of the stock process we have encountered are continuous, i.e the stock price is not supposed to have jumps. Very quickly we see this assumption is restrictive: when the stock pays dividend, the stock price has a downward jump corresponding to the amount of divident payout. However, the dividend payment can be covered within the continuous framework without introducing any new ideas, essentially because the dividend payment times are **deterministic**.

Dividend payments are not the only phenomenons that cause the stock price to have jumps, obviously. In reality, we quickly observe many instances where stock price jumps, and the most important characteristic of these jumps is that they happen at **random** times. Being able to model stock prices that incorporate jumps at random (or more precisly, stopping times) and learning how to price financial products based on these models are the main focus of this Chapter.

2 Overview of price modeling in continuous time

Let $\{\mathcal{F}(t);\, t \geq 0\}$ be a filtration modeling the accumulation of market information available to investors as time progresses. A simple paradigm guides the construction of models for an asset price, $\{S(t);\, t \geq 0\}$, that is a continuous function of time:

$$\frac{\mathrm{d}S(t)}{S(t)} = \alpha(t)\,\mathrm{d}t + dM(t), \tag{1}$$

where α is a process adapted to $\{\mathcal{F}(t);\, t \geq 0\}$, M is a martingale with respect to $\{\mathcal{F}(t);\, t \geq 0\}$, and $dS(t) = S(t + \mathrm{d}t) - S(t)$ denotes the price increment at t for an infinitesimally small positive increment of time, $\mathrm{d}t$. This is a formal equation, because $\mathrm{d}t$ is not precisely defined. Intuitively, (1) says that if $\mathrm{d}t$ is replaced by a small finite time, the left- and right-hand sides are approximately equal and that the approximation is better the smaller $\mathrm{d}t$ is.

This modeling framework is entirely natural. Because M is a martingale,

$$E[dM(t)\Big|\mathcal{F}(t)] = E[M(t+\mathrm{d}t) - M(t)\big|\mathcal{F}(t)] = 0.$$

Therefore

$$E\Big[\frac{\mathrm{d}S(t)}{S(t)}\Big|\mathcal{F}(t)\Big] = \alpha(t)\,\mathrm{d}t.$$

This means that $\alpha(t)\,\mathrm{d}t$ is the expected infinitesimal return on owning a share of the asset over the period $[t, t+\mathrm{d}t]$, conditional on the market history at time t. Therefore, from equation (1), $dM(t)$ is the fluctuation of the return about this conditional mean. Essentially, M is *the source of the random fluctuations in the price; we say that the noise M drives the evolution of S.* Using paradigm (1), we can break down price modeling into the separate problems of modeling α and M.

3 Models based on Brownian motion, a review

From the perspective of equation (1), the theory of stochastic integrals with respect to Brownian motion is a mechanism for producing a large and flexible class of martingales to use for M. Let us recall in broad outline, how this theory goes. We start with a continuous process W, namely Brownian motion, which is not just a martingale, but a process with independent and stationary increments. It is assumed that W is adapted to $\{\mathcal{F}(t);\, t \geq 0\}$ and that future increments $W(t+h) - W(t)$, $h \geq 0$ are independent of $\mathcal{F}(t)$. We then define $\int_0^t \alpha(s)\,\mathrm{d}W(s)$ for processes α adapted to $\{\mathcal{F}(t);\, t \geq 0\}$ and satisfying $E[\int_0^T \alpha^2(s)\,\mathrm{d}s] < \infty$ for all T. The definition proceeds in two steps. First, we define the integral if α has the form:

$$\alpha(t) = \sum_{i=1}^{t} \alpha_i \mathbf{1}_{(t_{i-1},t_i]}(t)$$

where $t_0 < t_1 < \cdots < t_n$ and α_i is $\mathcal{F}(t_{i-1})$-measurable for each i. The definition is:

$$\int_0^t \alpha(s)\,\mathrm{d}W(s) := \sum_{i=1}^{n} \alpha_i[W(t_i \wedge t) - W(t_{i-1} \wedge t)]$$

This is the accumumlated return, up to time t, from betting amount α_i on the increment of W over every interval $[t_{i-1} \wedge t, t_i \wedge t]$. We then show this integral satisfies the Itô isometry,

$$E\Big[\Big(\int_0^t \alpha(s)\,\mathrm{d}W(s)\Big)^2\Big] = E\Big[\int_0^t \alpha^2(s)\,\mathrm{d}s\Big],$$

use this isometry to extend the definition to more general, adapted integrands, and obtain in this way a large family of martingales of the form,

$$M(t) := \int_0^t \sigma(s)\,\mathrm{d}W(s),$$

where $\sigma(t)$ is adapted to $\{\mathcal{F}(t);\, t \geq 0\}$ and $E[\int_0^T \sigma^2(s)\,\mathrm{d}s] < \infty$ for all T.

When $\int_0^t \sigma(s)\,\mathrm{d}W(s)$ is used for M in (1), the price model becomes:

$$dS(t) = \alpha(t)S(t)\,\mathrm{d}t + S(t)\,\sigma(t)\mathrm{d}W(t). \tag{2}$$

Here we have expressed $dM(t)$ as $\sigma(t)\mathrm{d}W(t)$; formally it consists of a Gaussian term $dW(t)$, with mean zero and variance $\mathrm{d}t$, *independent* of the past, times a volatility factor $\sigma(t)$ that is known at time t. For this class of price equations, the job of modeling reduces to choosing α and σ. (Pricing derivatives requires taking expectations with respect to a risk-neutral measure, and we found that this measure does not depend on α. Therefore, for pricing we really only need to model volatility.)

4 The problem of modeling jumps

As we mentioned in the Introduction, the main constraint of model (2) above is continuity; $W(t)$, $M(t) = \int_0^t \sigma(s)\,\mathrm{d}W(s)$, and, consequently, the price $S(t)$ solving (2) are all **continuous functions of** t with probability one.

Of course, in reality *prices move in steps of discrete size.* So long as these steps are small, continuous models of form (2) should be okay, if returns over small intervals are approximately Gaussian. But occasionally, prices take large, sudden and unexpected jumps, such as a market shock, and returns might not be Gaussian.

Therefore, one would like to allow jumps in price models. This will not only incorporate the phenomenon of sudden large jumps, but will also offer a richer family of models for fitting the empirically observed, statistical behavior of prices.

Price models with jumps can be obtained by introducing jumps into the noise, M, in equation (1). To proceed we need to know **how to define martingales with**

jumps and we need **a theory to interpret and solve stochastic differential equations with jump terms.**

The strategy for carrying this out parallels stochastic integration theory for Brownian motion. One starts with *independent increment* processes, $X(t)$, that are martingales and that have jumps. The simplest example is *the compensated Poisson process.* Then one defines stochastic integrals, $\int_0^t \sigma(s)\,\mathrm{d}X(s)$, and **establishes conditions on $\sigma(\cdot)$ so that these integrals are martingales.** This produces a large class of martingales with jumps to use as the driving noise in price equations. Finally, one **extends Itô calculus to stochastic integrals with jumps.** This calculus can then be used to analyze derivatives based on the new price models.

The compensated Poisson process is derived from the Poisson process. To understand it, we start out with the Poisson process.

Remark 4.1. *As mentioned in the Introduction, we can also have stock price jumps in the case of dividend payments. In this case the stock will be modeled as followed:*

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) + S(t-)dJ(t),$$

where if we let $0 < t_1 < \cdots < t_n < T$ *be the dividend payment days and* $\alpha_i, i = 1, \cdots, n$ *be the dividend percentage (that is at time* t_i *the dividend paid is* $\alpha_i S(t_i-)$*) then*

$$J(t) = \sum_{i=1}^{n} -\alpha_i \mathbf{1}_{\{t \geq t_i\}}.$$

(See also Shreve's Section 5.5.3)

The point is here $J(t)$ *is NOT a martingale, nor can it be made into a martingale by being compensated as a compensated Poisson process. Thus the dividend payment stock's model does not fall into the theory of jump processes discussed in this Chapter as far as the martingale aspect is concerned. However, the mathematical tools developed here can be used to analyze the jump part in the dividend paying stock in a similar way as we present later in this Chapter.*

5 The most basic model of jump processes: Poisson process

5.1 Heuristics about Poisson process

We think of Poisson process as followed: suppose that we have an alarm clock that will ring after a *random* time τ, where τ is exponentially distributed with some mean $\frac{1}{\lambda}$. We keep account of the value of the Poisson process at any time t by the notation $N(t)$. At time 0, we set the alarm clock and set $N(0) = 0$. When the alarm rings, we increase the value of N by 1, that is we set $N(\tau) = 1$ and repeat the whole process (i.e. we reset the alarm clock and increase the value of N by 1 the next time the clock rings). The resulting process $N(t)$ is then a Poisson process with rate λ. We observe that the larger λ is, the clock would be likely to ring sooner and the more jumps would likely happen in a given time interval [0,T]. It is also clear that $N(t)$ is constant in between the "ring" times.

5.2 Formal mathematical definition

a. τ (as a R.V.) is said to be exponentially distributed with rate λ if it has the density

$$f(t) = \lambda e^{-\lambda t}\mathbf{1}_{(t\geq 0)}.$$

It follows that $E(\tau) = \frac{1}{\lambda}$ and $Var(\tau) = \frac{1}{\lambda^2}$. An important property of exponential random variable is the *memoryless property*:

$$\mathbb{P}(\tau > t + s|\tau > s) = \mathbb{P}(\tau > t).$$

b. Let $\tau_i, i = 1, 2, ...$ be a sequence of i.i.d. Exponential(λ). Let $S_k := \sum_{i=1}^{k} \tau_i$. The Poisson process $N(t)$ with rate λ is defined as:

$$N(t) = \sum_{i=1}^{\infty} \mathbf{1}_{(t\geq S_i)}.$$

τ_i is called the *inter-arrival time.* It is the wait time from the $(i-1)^{th}$ jump to the i^{th} jump. S_i is called the *arrival time.* It is the time of the i^{th} jump.

5.3 Important basic properties

a. Distribution: $N(t)$ is has distribution Poisson(λt), that is

$$\mathbb{P}(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

Proof. Let $S_n = \sum_{i=1}^n \tau_i$ be the arrival time, then

$$\begin{aligned}\mathbb{P}(N(t)=k) &= \mathbb{P}(S_{k+1} > t, S_k \le t) \\ &= \mathbb{P}(S_{k+1} > t) - \mathbb{P}(S_{k+1} > t, S_k > t) = \mathbb{P}(S_{k+1} > t) - \mathbb{P}(S_k > t).\end{aligned}$$

From Shreve's Lemma 11.2.1, S_n has Gamma(λ, n) distribution. That is, it has the density:

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!}\lambda e^{-\lambda s}, s \ge 0.$$

It is a straight forward matter of integration now to verify that

$$\mathbb{P}(S_{k+1} > t) - \mathbb{P}(S_k > t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

The integration can be tedious, however. Another way to verify it is as followed: Denote $f(t) := \mathbb{P}(S_{k+1} > t) - \mathbb{P}(S_k > t)$ and note that $f(t)$ satisfies the following ODE:

$$\begin{aligned} f'(t) &= g_k(t) - g_{k+1}(t) = \frac{(\lambda t)^{k-1}}{(k-1)!}\lambda e^{-\lambda t} - \frac{(\lambda t)^k}{k!}\lambda e^{-\lambda t} \\ f(0) &= 0. \end{aligned}$$

It is clear that $f(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$ is the *unique* solution to the above ODE. The verification is complete. ∎

b. $N(t)$ has independent increment. That is if we denote $\mathcal{F}_t$ to be the filtration generated by $N(s), 0 \le s \le t$ then for all $t \le t_1 < t_2$, $N(t_2) - N(t_1)$ is independent of $\mathcal{F}_t$.

Heuristic reason: Let $0 \le s < t$. Clearly $N(t) - N(s)$ counts the number of jumps starting from time s. Given all the information up to time s, what is the distribution of the first jump time after s? That is, we want to compute $\mathbb{P}(S_{N(s)+1} \ge t | \mathcal{F}_s)$, where S_n is the arrival time as defined in Shreve (11.2.4). Note that since $N(s)$ represents the number of jumps up to time s, $S_{N(s)+1}$ is exactly the time of *the first jump after time s.*

But this is the same as computing $\mathbb{P}(\tau_{N(s)+1} \ge t - S_{N(s)} | \tau_{N(s)+1} \ge s - S_{N(s)})$. Note that $S_{N(s)}$ here represents the time of the *last jump before time s,* and $\tau_{N(s)+1}$ is *the wait time between the last jump before time s and the first jump after time s.* So $\mathbb{P}(\tau_{N(s)+1} \ge t - S_{N(s)} | \tau_{N(s)+1} \ge s - S_{N(s)})$ asks for the probability that we have to wait until after time t for the first jump after time s, given that we know we have waited up until time s since the last jump before s, which has the same content as $\mathbb{P}(S_{N(s)+1} \geq t|\mathcal{F}_s)$.

Note also that $\mathbb{P}(\tau_{N(s)+1} \geq t - S_{N(s)}|\tau_{N(s)+1} \geq s - S_{N(s)}) = \mathbb{P}(\tau_{N(s)+1} \geq t - s + s - S_{N(s)}|\tau_{N(s)+1} \geq s - S_{N(s)})$. Since $\mathcal{F}_s$ is given, $N(s)$ should be looked at as a constant here. But from the memoryless property of $\tau_{N(s)+1}$, we get

$$\mathbb{P}(\tau_{N(s)+1} \geq t - s + s - \tau_{N(s)}|\tau_{N(s)+1} \geq s - \tau_{N(s)}) = \mathbb{P}(\tau_{N(s)+1} \geq t - s).$$

That is, the first jump time after s can be looked at as an exponential clock starts at time s, hence independent of the past information. Using the independence of inter-arrival times, it is clear now that the increments of $N(t)$ after time s is independent of the information up to time s. ■

c. $N(t)$ has stationary increment. More specifically, $N(t) - N(s)$ has distribution Possion($\lambda(t-s)$).

Heuristic reason: It follows from the same arguments of part b. ■

6 Generalizations of Poisson process

6.1 Compound Poisson process

The Poisson process we introduced has the satisfactory property that it jumps at *random* times. However, each of the jump is by definition of length 1, which is rather restrictive. It is desirable in terms of being realistic to have random jumps in our model. To that end, we proceed as followed.

Let $N(t)$ be a Poisson process with rate λ and let $Y_0 = 0, Y_i, i = 1, 2, ...$ be i.i.d.(and also independent of N(t)) with $E(Y_i) = \mu$. Define

$$Q(t) = \sum_{i=0}^{N(t)} Y_i,$$

then $Q(t)$ is called a compound Poisson process. Similar to a Poisson process, $Q(t)$ also has the basic properties of *independent and stationary increments.* We *do not know* the specific distribution of $Q(t) - Q(s)$ (it depends on the distribution of Y_i's , of course), but we do know that $E(Q(t) - Q(s)) = \mu\lambda(t - s)$.

6.2 Pure jump process

Poisson process and compound Poisson process are examples of pure jump processes.

Definition 6.1. *A stochastic process* $\{J(t); t \geq 0\}$ *is called a* **pure jump process** *if its sample paths are* **right-continuous** *and* **piecewise-constant**. *(Recall that this entails that each sample path of* $J(t)$ *admits only a finite number of jumps in any finite time interval.) A pure jump process* N *is called a* **counting process** *if* $N(0) = 0$ *and it its jumps* **all have magnitude 1;** *hence it can only increase, and is always integer-valued. If* N *is a counting process,* $N(t)$ *counts the number of jumps in the time interval* $[0, t]$.

Remark 6.2. *In stochastic integration theory, the definition of a pure jump process is more general than the one here and allows infinite numbers of jumps in finite time intervals.*

Remark 6.3. *By right continuity, a pure jump process* $J(t)$ *CANNOT jump at time 0, which is always the conventional time that we start observing the process.*

6.3 Levy process

So far the three processes that we have encountered: Brownian motion, Poisson and compound Poisson processes have these three properties in common:

- Its value at time 0 is 0 : $X(0) = 0$.
- It has càdlàg path.
- It has stationary and independent increments.

A process $X(t)$ is said to be a Levy process *starting at 0* if it satisfies these three properties (clearly if we change the first property to $X(0) = x$ then we would get a Levy process starting at x). Brownian motion is an example of a continuous Levy process and Poisson process is an example of a pure jump Levy process. Indeed, Brownian motion, compound Poisson process and pure jump process may be thought as "building blocks" of a Levy process (See Levy-Ito decomposition on Wikipedia, for example).

A rather simple but important property of Levy process is as followed: If $X_1, X_2, ...X_n$ are *independent* Levy process then $\sum_{i=1}^{n} X_i$ is a Levy process. In particular, if we consider $S(t) = X(t) + Q(t)$, where $X(t)$ is a Geometric Brownian motion with the drift μ and volatility σ *constant*), $Q(t)$ a compound Poisson process then $S(t)$ is a Levy process.

7 Martingale property

In Math finance, we always require the discounted underlying to be a martingale, so that no arbitrage can happen. As mentioned, the Levy process is intimately connected to our stock models, so it's natural to first study the martingale property of Levy processes.

7.1 Levy process

Let $X(t)$ be a Levy process and $\mathcal{F}(t)$ its filtration. If $E(X(1)) = \mu$ then it can be shown that $E(X(t)) = \mu t$. Similarly, if $Var(X(1)) = \sigma^2$ then it can be shown that $Var(X(t)) = \sigma^2 t$. Since X has independent increment, one can check that

$$\begin{aligned} Y(t) &= X(t) - \mu t; \\ Z(t) &= (X(t) - \mu t)^2 - \sigma^2 t \end{aligned}$$

are martingales with respect to $\mathcal{F}(t)$.

7.2 Brownian motion

Let $W(t)$ be a Brownian motion and $\mathcal{F}(t)$ its filtration. Then $W(t)$ and $W^2(t) - t$ are martingales w.r.t. $\mathcal{F}(t)$. More importantly, we have the following exponential martingale associated with Brownian motion:

$$Z(t) = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}.$$

7.3 Poisson process

Let $N(t)$ be a Poisson process and $\mathcal{F}(t)$ its filtration. Then $N(t) - \lambda t$ (called a *compensated Poisson process*) and $(N(t) - \lambda t)^2 - \lambda t$ are martingales w.r.t. $\mathcal{F}(t)$. We also have the following exponential martingale associated with $N(t)$:

$$Z(t) = \exp\big(iuN(t) - \lambda t(e^{iu} - 1)\big), \forall u \in \mathbb{R}.$$

7.4 Compound Poisson process

Let $Q(t)$ be a compound Poisson process and $\mathcal{F}(t)$ its filtration. That is

$$Q(t) = \sum_{i=0}^{N_t} Y_i,$$

where $N(t)$ is a Possion(λ) process. Let $E(Y_1) = \mu$ and $Var(Y_1) = \sigma^2$. One can check that

$$E(Q_t) = \lambda\mu t$$

and

$$Var(Q_t) = \lambda t(\sigma^2 + \mu^2).$$

Then

$$Q(t) - \mu\lambda t$$

(called a *compensated compound Poisson process*) and

$$(Q(t) - \mu\lambda t)^2 - \lambda t(\sigma^2 + \mu^2)$$

are both martingales w.r.t. $\mathcal{F}(t)$.

Let $\phi(u) := \mathbb{E}(e^{iuY_1})$ be the characteristic function of Y_i. Then we also have the following exponential martingale associated with $Q(t)$:

$$Z(t) = \exp\big(iuQ(t) - \lambda t(\phi(u) - 1)\big), \forall u \in \mathbb{R}.$$

8 Lebesgue-Stieltjes integral

8.1 Motivation

Now that we have introduced Poisson process, it is easy to see how to incorprate jumps into the current Black-Scholes stock model. Specifically, let $X(t)$ be a geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

and $N(t)$ a Poisson process. Then defining the stock process as $S(t) := X(t) + N(t)$ already gives us a stock price that jumps at random times, and in between the jumps behave as a geometric Brownian motion.

Let Δ_t represent the number of shares of $S(t)$ we hold at time t. As you might remember from the previous material, we need to know how to evaluate the integral $\int_0^t \Delta_s dS_s$, since it is connected with the value of a portfolio that has S as a component. It is reasonable to expect that

$$\int_0^t \Delta_s dS_s = \int_0^t \Delta_s dX_s + \int_0^t \Delta_s dN_s,$$

and we already know how to evaluate $\int_0^t \Delta_s dX_s$ from the chapter on Ito integral. It remains to define $\int_0^t \Delta_s dN_s$.

For each event ω, the path $N_t(\omega)$ (as a function of t) belongs to a special class of functions called *the functions of bounded variation.* For this reason, $\int_0^t \Delta_s dN_s$ is defined via the concept of Lebesgue-Stieltjes integral of classical analysis. It still has some subtleties, however, mostly due to the facts that $N(t)$ has *jumps*, so the regularity (left or right continuity) of the integrand Δ_t affects the value of the integral. For this reason, we will review some basic aspects of the Lebesgue-Stieltjes integral with respect to càdlàg integrator in the next section.

8.2 Mathematical preliminaries; right-continuous functions with left limits

1. *Limits, continuity and jumps.* Let $f(t)$ be a function defined for $t \geq 0$. The right limit of f at $t \geq 0$ is

$$f(t+) := \lim_{s \downarrow t} f(s), \quad \text{assuming it exists.}$$

The left limit of f at $t > 0$ is

$$f(t-) := \lim_{s \uparrow t} f(s), \quad \text{assuming it exists.}$$

As a convention, we set $f(0-) = f(0)$. The jump of f at t is the difference of these limits and is denoted

$$\triangle f(t) = f(t+) - f(t-)$$

A function f is said to be right-continuous with left limits if $f(t) = f(t+)$ for all t and if $f(t-)$ exists for all t. Such functions are sometimes called *càdlàg* functions in the literature. It is worth knowing this term so we shall use it. It is an acronym of the French phrase meaning 'right-continuous with left limits': *continu à droite, limites à gauche.*

If we are to allow jumps in price models, we need a convention for *what the price is at the exact time of the jump.* Our convention shall be that all prices are càdlàg functions. Hence $S(t) = S(t+)$ is the price that the asset jumps to at time t, $S(t-)$ is the price immediately before the jump, and $\triangle S(t) = S(t) - S(t-)$ is the size of the jump.

The same convention will be imposed on the driving martingale M in our price models, since it is the martingale M that causes the jump. Likewise, in the theory of stochastic integration that we develop, *stochastic integrals will have càdlàg paths.*

Along with these conventions we need to re-interpret the intuitive meaning of $dS(t)$. You should think of this as the increment $S(t+\mathrm{d}t) - S(t-)$; Of course, this coincides with $S(t+\mathrm{d}t)-S(t)$ at times t at which $S(t)$ is continuous; if not, *it captures the jump at time t.* Note that, as usual, the identity, $dS(t) = S(t+\mathrm{d}t) - S(t-)$, does not have a strict meaning, since $\mathrm{d}t$ does not have a strict meaning, but it is a correct guide to thinking about the movements of S over small time intervals.

2. *Important facts to know about càdlàg functions.* Let f be càdlàg:

(i) The function $t \to f(t-)$ is left-continuous.

(ii) The set of points at which f is *not* continuous is either finite or countably infinite.

(iii) Since $f(t)$ and $f(t-)$ differ only at points at which f is not continuous and there are only countably many such points, $\int_0^T f(t-)g(t)\,\mathrm{d}t = \int_0^T f(t)g(t)\,\mathrm{d}t$, for any $T > 0$ and g.

8.3 Lebesgue-Stieltjes integrals for increasing, right-continuous integrators

A. Let $G(t)$, $t \geq 0$, be a function that can be written in the form

$$G(t) = A_1(t) - A_2(t), \text{ where } A_1 \text{ and } A_2 \text{ are } \textit{increasing and right-continuous.} \quad (3)$$

Since A_1 and A_2 are increasing, $A_i(t-) = \lim_{s\uparrow t} A_i(s)$ exists automatically for all $t > 0$, and for each $i = 1, 2$. Hence A_1, and A_2, and, therefore, G, are all càdlàg.

It is easy to see that if $G_1(t)$ and $G_2(t)$ are both functions with the property (3), then so is any linear combination $\alpha G_1(t) + \beta G_2(t)$.

A function G that can be written as the difference of two increasing function has the special property of being a function of *bounded variation.* You will learn what this means in a homework exercise. Conversely, any function of bounded variation may be written as the difference of increasing functions. Therefore, we shall summarize the condition (2) by saying G is a càdlàg function of bounded variation.

Examples of functions of the form (3).

(i) G is a right-continuous, piecewise constant functions of the form

$$G(t) = \sum_0^n a_k \mathbf{1}_{[t_k, t_{k+1})}(t)$$

where $0 = t_0 < t_1 < \cdots < t_n$, and $t_{n+1} = \infty$.

(ii) $G(t)$ is a differentiable function, $G(t) = G(0) + \int_0^t g(s)\,\mathrm{d}s$.

(iii) $G(t) = \int_0^t g_1(s)\,\mathrm{d}s + G_2(t)$, where G_2 is right-continuous and piecewise constant, as in (i).

Explanation : (i) The function $\mathbf{1}_{[a,\infty)}(t)$ is right-continuous and increasing. Therefore $\mathbf{1}_{[a,b)}(t) = \mathbf{1}_{[a,\infty)}(t) - \mathbf{1}_{[b,\infty)}(t)$ is a function of form (3). Because G as defined in (i) is a linear combination of functions of the form (3), it also has this form.

As for (ii), observe that

$$G(t) = G(0) + \int_0^t g(s)\,ds = G(0) + \int_0^t |g(s)| \mathbf{1}_{\{g(s)\geq 0\}}\,\mathrm{d}s - \int_0^t |g(s)| \mathbf{1}_{\{g(s)<0\}}\,\mathrm{d}s$$

decomposes G into the difference of two continuous, increasing functions.

(iii) is a consequence of (i) and (ii). $\diamond$

B. If G is a càdlàg function of bounded variation, there is a natural way to define integrals, which we shall denote,

$$\int_0^t H(s)\,\mathrm{d}G(s),$$

built on the increments of G. In the mathematical literature, these are called Lebesgue-Stieltjes integrals.

8.4 Lebesgue-Stieltjes integral for left-continuous, piecewise constant integrands

A function $H(t)$, $t \geq 0$ is left-continuous and piecewise constant if it has the form:

$$H(s) = \sum_{i=1}^n c_i \mathbf{1}_{(t_{i-1}, t_i]}(s) + c_{n+1} \mathbf{1}_{(t_n,\infty)}(s) \tag{4}$$

where $0 = t_0 < t_1 < \cdots < t_n$. For convenience let $t_{n+1} = \infty$. For this H, define

$$\int_{(0,t]} H(s)\,\mathrm{d}G(s) := \sum_{i=1}^{n+1} c_i \Big[G(t_i \wedge t) - G(t_{i-1} \wedge t) \Big] \tag{5}$$

Usually, we will write the integral in (5) as $\int_0^t H(s)\,\mathrm{d}G(s)$.

Motivation and comments.

1. This definition should be no surprise; the idea is to multiply the value of H on each interval $(t_{i-1}, t_i]$ by the increment of G over that interval. If $G(s) = s$, then

$$\int_{(0,t]} H(s)\,\mathrm{d}G(s) = \sum_{i=1}^{n+1} c_i \Big[t_i \wedge t - t_{i-1} \wedge t\Big] = \int_0^t H(s)\,\mathrm{d}s,$$

which is the usual Riemann (or Lebesgue) integral. If instead we replaced G by a Brownian motion W, we would get the Itô integral of H.

2. The fact that we used intervals of the form $(t_{i-1}, t_i]$ in the definition of H in (5), so that H is left-continuous, **is not an accident and is tied up with the assumption that G is right-continuous**.

First, we model G as being right-continuous according to our intuition that a shock cannot be predictable (you can observe the behavior of the stock up until the time of the shock - the jump time - but you will not be able to tell the value of the stock after the jump based on your observation).

Second, the result in Example 1 below also works well with our intuition: the change in the portfolio value after the shock is the change in the stock price ($\Delta G(\tau)$) multiplied with the number of shares we hold at the time of the shock ($H(\tau)$). Note that if we use a right continuous integrand $H(s)$, we will NOT get a similar result. This explains the choice of left continuous integrand for our basic building block of Lebesgue-Stieltjes integral w.r.t. right continuous integrator.

Example 1. Consider the simplest example, where G is piecewise constant with a single jump at time τ:

$$G(t) = \begin{cases} a_0, & \text{if } 0 \leq t < \tau; \\ a_1, & \text{if } t \geq \tau, \end{cases} \tag{6}$$

where $a_0 \neq a_1$. We will show that

$$\int_{(0,t]} H(s)\,\mathrm{d}G(s) = \begin{cases} H(\tau)\triangle G(\tau), & \text{if } t \geq \tau; \\ 0, & \text{if } t < \tau. \end{cases} \qquad \diamond \tag{7}$$

Let H be given as in (4). Observe that

$$G(t_k \wedge t) - G(t_{k-1} \wedge t) = \begin{cases} a_1 - a_0 = \triangle G(\tau), & \text{if } t_{k-1} \wedge t < \tau \leq t_k \wedge t; \\ 0, & \text{otherwise.} \end{cases}$$

Also, notice that if $t_{k-1} < \tau \leq t_k$, then $H(\tau) = c_k$, since H has the constant value c_k on $(t_{k-1}, t_k]$. Thus each term in the sum on the right-hand side of (5) is zero, except if $t_{k-1} \wedge t < \tau \leq t_k \wedge t$, and for this k, $c_k \Big[G(t_k \wedge t) - G(t_{k-1} \wedge t)\Big] = H(\tau)\triangle G(\tau)$. If $t < \tau$, there is no k such that $t_{k-1} \wedge t < \tau \leq t_k \wedge t$, and so the sum in (5) is zero. If $t \geq \tau$, the sum contains one non-zero term, which we have shown equals $H(\tau)\triangle G(\tau)$. This proves (7).

Example 2. Let $G(t) = G(0) + \int_0^t g(s)\,\mathrm{d}s$, that is, G is differentiable and $G'(t) = g(t)$. Let H be as in (4). Then

$$\begin{aligned}\int_{(0,t]} H(s)\,\mathrm{d}G(s) &= \sum_{i=1}^{n+1} c_i \Big[G(t_i \wedge t) - G(t_{i-1}) \wedge t)\Big] \\ &= \sum_{i=1}^{n+1} c_i \cdot \int_{t_{i-1}\wedge t}^{t_i \wedge t} g(s)\,\mathrm{d}s \\ &= \int_0^t \Big[\sum_{i=1}^{n+1} c_i \mathbf{1}_{(t_{i-1},t_i]}(s)\Big]\, g(s)\,\mathrm{d}s \\ &= \int_0^t H(s) g(s)\,\mathrm{d}s, \qquad \diamond \qquad (8)\end{aligned}$$

8.5 Lebesgue-Stieltjes integral for Borel measurable integrands

The following theorem states that definition (5) can be extended in a unique way to a large class of integrands. The proof requires tools of measure theory beyond the scope of this course. It is only important for you to know what the theorem says. This will usually be enough for you to understand what is going on if you encounter Lebesgue-Stieltjes integrals when reading mathematical finance literature.

Theorem 1. *There is a* **unique** *way to assign to each bounded, Borel measurable function H and bounded variation function G, an integral $\int_0^t H(s)\,\mathrm{d}G(s)$ for $t > 0$, with the following properties:*

(i) $\int_0^t H(s)\,\mathrm{d}G(s)$ *is defined by (5) when H has the form given in (4);*

(ii) (linearity) $\int_0^t [a_1 H_1(s) + a_2 H_2(s)]\,\mathrm{d}G(s) = a_1 \int_0^t H_1(s)\,\mathrm{d}G(s) + a_2 \int_0^t H_2(s)\,\mathrm{d}G(s)$*; and*

$$\int_0^t H(s)\,\mathrm{d}\big[aG_1(s) + bG_2(s)\big] = a\int_0^t H(s)\,\mathrm{d}G_1(s) + b\int_0^t H(s)\,\mathrm{d}G_2(s);$$

(iii) (exchange of limit and integral) Assume $H(s) = \lim_{n\to\infty} H_n(s)$ for all s, where, for some $K < \infty$, $\big|H_n(s)\big| \leq K$ for all n and $s \geq 0$. Then

$$\int_0^t H(s)\,\mathrm{d}G(s) = \lim_{n\to\infty} \int_0^t H_n(s)\,\mathrm{d}G(s).$$

Unfortunately, Theorem 1 only assures us that the Lebesgue-Stieltjes integral can be defined in a meaningful way; it does not directly say how to compute one. Fortunately, in most situations we shall encounter, the Lebesgue-Stieltjes integral can be reduced to familiar and easy-to-handle objects.

8.5.1 $G(t)$ is continuously differentiable

When $G(t) = G(0) + \int_0^t g(s)\,ds$, and H is *any* bounded, Borel function

$$\int_0^t H(s)\,\mathrm{d}G(s) = \int_0^t H(s)g(s)\,\mathrm{d}s. \tag{9}$$

We saw this is true in Example 2, when H is a left-continuous, piecewise constant function. This shows that the right-hand side of (9) satisfies property (i) of Theorem 1. It also satisfies the properties in (ii), as one can show by direct calculation, and it satisfies property (iii) because of the properties of the Legesgue integral (full explanation omitted!). Thus $\int_0^t H(s)g(s)\,\mathrm{d}s$ must coincide with $\int_0^t H(s)\,\mathrm{d}G(s)$, because, by Theorem 1, the latter integral is uniquely determined by properties (i)—(iii).

8.5.2 $G(t)$ is a pure jump function

Let G be piecewise-constant of the form

$$G(t) = a_0\mathbf{1}_{[0,\tau_1)}(t) + a_1\mathbf{1}_{[\tau_1,\tau_2)}(t) + \cdots + a_n\mathbf{1}_{[\tau_{n-1},\tau_n)} + \cdots$$

Thus G is constant except for jumps at the points $\tau_1 < \tau_2 < \cdots$. Notice that G is defined so as to be càdlàg.

In this case, also for *any* bounded, Borel function H we have

$$\int_0^t H(s)\,\mathrm{d}G(s) = \sum_{j;\tau_j \le t} H(\tau_j)\triangle G(\tau_j). \tag{10}$$

To emphasize, the sum is over the jump times of G that occur at time t or before. Since $\triangle G(s) = 0$ if s is not equal to any jump time, it is convenient to write this formula as

$$\int_0^t H(s)\,\mathrm{d}G(s) = \sum_{0<s\le t} H(s)\triangle G(s). \tag{11}$$

This result is derived from Theorem 1 by showing that the expression on the right-hand side of (10) satisfies properties (i)—(iii) of the Theorem. For simplicity, consider the case when G jumps only at a finite number of times. Properties (ii) and (iii) are easy to verify directly. In example 1, we have verified property (i) for the case when G has a single jump and H is a left-continuous, piecewise constant function. Thus, Theorem 1 implies that (10) is true if G has just one jump. If G has multiple jump times, we can write $G(s) = G_1(s) + G_2(s) + \cdots + G_n(t)$, where each G_i is a piecewise-constant, càdlàg function jumping at only one time, and use property (ii) of Theorem (1) to deduce (10).

8.5.3 Combination of the above two cases

We will encounter the case $G(t) = \int_0^t g_1(s)\,\mathrm{d}s + G_2(s)$, where G_2 is piecewise constant, càdlàg, as in (8.5.2). Then, by property (ii) of Theorem 1,

$$\int_0^t H(s)\,\mathrm{d}G(s) = \int_0^t H(s)g_1(s)\,\mathrm{d}s + \sum_{0<s\leq t} H(s)\triangle G_2(s).$$

However, notice that $\triangle G(s) = \triangle G_2(s)$ for all s because the integral term in G is continuous. Therefore we can rewrite the formula as

$$\int_0^t H(s)\,\mathrm{d}G(s) = \int_0^t H(s)g_1(s)\,\mathrm{d}s + \sum_{0<s\leq t} H(s)\triangle G(s). \tag{12}$$

If the jumps of G occur at times $0 < \tau_1 < \tau_2 < \ldots$, then $G'(s) = g_1(s)$ exists at any time s not equal to a jump time. Thus,

$$\int_0^t H(s)\,\mathrm{d}G(s) = \sum_i \int_{\tau_{i-1}\wedge t}^{\tau_i\wedge t} H(s)G'(s)\,\mathrm{d}s + \sum_{s\leq t} H(s)\triangle G(s). \tag{13}$$

This is the easiest form to use.

Because of item (iii) on page 4, if G is differentiable and $G'(s) = g(s)$, and if H is cádlàg, $\int_0^t H(s-)\,\mathrm{d}G(s) = \int_0^t H(s-)g(s)\,\mathrm{d}s = \int_0^t H(s)g(s)\,\mathrm{d}s = \int_0^t H(s)\,\mathrm{d}G(s)$. But if G has jumps, the two integrals may not agree, as we shall see by example later.

Remark Let $\{W(t)(\omega);\, t \geq 0, \omega \in \Omega\}$ be a Brownian motion. The definition of stochastic integrals with respect to W was a fairly complicated affair. We did need such a complex definition? Why not just define $\int_0^t H(s)\,\mathrm{d}W(s)$ by applying Theorem 1 to $W(\cdot)(\omega)$ for each ω and be done with it? Or did we use a complicated definition only because of some clandestine conspiracy to make the lives of math finance students miserable? Solemn vows of secrecy forbid me from answering the last question, but the first two are easy to answer. We cannot apply Theorem 1 to Brownian motion because, with probability one, the paths of Brownian motion *are not functions of bounded variation.* This is due to the fact that Brownian motion has non-trivial quadratic variation. You will get to explore this point in a homework exercise. It is absolutely essential to your understanding of stochastic integration.

8.6 Stochastic Integration

Let $\{X(t)(\omega);\, t \geq 0, \omega \in \Omega\}$ be a stochastic process defined on a probability space $(\Omega, I\!P)$. Assume that for every ω, $X(t,\omega)$ is a bounded variation function, as a function of t. Let $\{\alpha(t)(\omega);\, t \geq 0, \omega \in \Omega\}$ be another stochastic process. Then,

$$\int_0^t \alpha(s)(\omega)\,\mathrm{d}X(s)(\omega)$$

will always represent the Lebesgue-Stieltjes integral of the process α with respect to the process X. Usually, we suppress the explicit dependence on ω and simply write the integral as $\int_0^t \alpha(s)\,\mathrm{d}X(s)$.

9 Stochastic integration w.r.t. semi-martingales

9.1 Definition and examples

Let $X(t) = \int_0^t \gamma(s)dW_s + A(t)$, where $W(t)$ is a Brownian motion with respect to a filtration $\mathcal{F}(t)$, $\gamma(t) \in \mathcal{F}(t)$ be such that $\int_0^t \phi(s)dW_s$ is defined and $A(t) \in \mathcal{F}(t)$ a process of bounded variation. $X(t)$ is called a semi-martingale w.r.t. $\mathcal{F}(t)$.

Definition 9.1. *Let $\phi(t) \in \mathcal{F}(t)$ be so that $\int_0^t \phi(s)\gamma(s)dWs$ and $\int_0^t \phi(s)dA(s)$ are defined. Then we define*

$$\int_0^t \phi(s)dX(s) := \int_0^t \phi(s)\gamma(s)dWs + \int_0^t \phi(s)dA(s).$$

It is important to note here that $\int_0^t \phi(s)\gamma(s)dWs$ is an Ito integral, which is not defined path-wise (since $W(t)$ has infinite variation) and $\int_0^t \phi(s)dA(s)$ is a Lebesgue-Stieltjes integral, which is defined pathwise using the definition of Section (8).

Example 9.2. *(i) Let $X(t)$ be a compensated compound Poisson process, i.e. $X(t) = Q(t) - \lambda\mu t$ where $Q(t)$ is a compound Poisson process. Let S_k be the jump times of $Q(t)$. Then*

$$\int_0^t \phi(s)dX(s) = \sum_i \phi(S_i)Y_i\mathbf{1}_{(S_i \le t)} - \int_0^t \lambda\mu\phi(s)ds.$$

(ii) Let $X(t) = W(t) + J(t)$, where $J(t)$ is a pure jump procress. Then

$$\int_0^t \phi(s)dX(s) = \int_0^t \phi(s)dWs + \sum_{0<s\le t} \phi(s)\Delta J(s).$$

We understand the term $\sum_{0<s\le t}\phi(s)\Delta J(s)$ as followed: for each event ω, let $0 < t_1(\omega) < t_2(\omega) < ... < t_{n(\omega)}(\omega) \le t$ be the jump times of $J(t)$. (The fact that there are finitely many jumps in $[0,t]$ and there is no jump at $t = 0$ come from the definition of pure jump process). Also note that the number of jumps in $[0,t], n(\omega)$ is random. Then

$$\int_0^t \phi(s)dJ(s)(\omega) = \sum_{0<s\le t} \phi(s)\Delta J(s) = \sum_{i=1}^{n(\omega)} \phi(t_i)[J(t_i) - J(t_i-)](\omega).$$

9.2 Martingale properties

Suppose we model our stock as

$$S(t) = \sigma W(t) + X(t),$$

where $W(t), X(t) \in \mathcal{F}(t)$ are independent, $W(t)$ is a Brownian motion and $X(t) = Q(t) - \lambda\mu t$ is a compensated compound Poisson process. Then $S(t)$ is a martingale. It is important for us then that if we denote $\phi(t)$ as the number of shares of S we hold at time t , $\int_0^t \phi(r)dS_r$ **is a martingale.**

From Ito integration, we know that if ϕ is an adapted process, then $\int_0^t \phi(s)dW(s)$ is a martingale. So it remains to ask if $\int_0^t \phi(s)dX(s)$ is also a martingale. However, this is not always the case. See Shreve's examples 11.4.4 and 11.4.6.

A suffficient condition for the stochastic integral w.r.t. a jump process (that is also a martingale) to be a martingale is that the integrand is left-continuous (and of course adapted). This is stated in Shreve's theorem 11.4.5. More generally, one can use a predictable integrand (a process that is the limit of a sequence of left-continuous processes) and the stochastic integral w.r.t. a jump martingale will still be a martingale.

In Shreve's example 11.4.6, the following process is considered:

$$\begin{aligned} X(t) &= \int_0^t \mathbf{1}_{[0,S_1]}(s)d(N(s)-\lambda s) \\ &= \int_0^t \mathbf{1}_{[0,S_1]}(s)dN(s) - \int_0^t \mathbf{1}_{[0,S_1]}(s)\lambda ds, \end{aligned}$$

where S_1 is the first jump time of $N(t)$. Note that the integrand here is left continuous.

For $t < S_1$, the integrand $\mathbf{1}_{[0,S_1]}(s) = 0$. Thus $X(t) = -\lambda t$.

For $t = S_1$, $\int_0^t \mathbf{1}_{[0,S_1]}(s)dN(s) = 1\Delta N(S_1) = 1$ while $\int_0^t \mathbf{1}_{[0,S_1]}(s)\lambda ds = \lambda S_1$. Thus $X(t) = 1 - \lambda S_1$.

For $s > S_1, \mathbf{1}_{[0,S_1]}(s) = 0$ thus $X(t) = 1 - \lambda S_1, t \geq S_1$.

We conclude that

$$\begin{aligned} X(t) &= -\lambda t \mathbf{1}_{(t<S_1)} + (1-\lambda S_1)\mathbf{1}_{(t\geq S_1)} \\ &= N(t\wedge S_1) - \lambda(t \wedge S_1). \end{aligned}$$

Here we can use the fact that a stopped martingale is a martingale to conclude that $X(t)$ is a martingale since $N(t) - \lambda t$ is a martingale and the above formula showed that $X(t)$ is a stopped martingale.

Using a similar argument, we have

$$Y(t) = \int_0^t \mathbf{1}_{[0,S_1)}(s)d(N(s)-\lambda s) = -\lambda(t\wedge S_1).$$

Heuristically, $\mathbb{P}(S_1 > 0) = 1$ therefore, for $s < t$,

$$\begin{aligned} \mathbb{P}\big(-\lambda(t\wedge S_1) \leq -\lambda(s\wedge S_1)\big) &= 1; \\ \mathbb{P}\big(-\lambda(t\wedge S_1) < -\lambda(s\wedge S_1)\big) &> 0. \end{aligned}$$

Therefore $\mathbb{E}(-\lambda(t\wedge S_1)) < \mathbb{E}(-\lambda(s\wedge S_1))$ and $Y(t)$ is not a martingale. A rigorous proof is provided in Shreve's.

10 Ito's formula for jump processes

10.1 Ito's formula for one jump process

The most general jump process we will consider in this chapter has the following form:

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \gamma(s)dWs + J(t),$$

where $J(t)$ is a pure jump process (Discussed in Section (6.2)). We also denote by $X^c(t)$ the continuous part of X, that is

$$X^c(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \gamma(s)dWs.$$

Given a function $f \in C^2$, we would like to obtain a formula for $df(X(t))$. We have the following observations:

(i) If $X(t) = X^c(t)$, i.e. if X has no jump then we have the classical Ito's formula:

$$df(X(t)) = f'(X(t))dXt + \frac{1}{2}f''(X(t))\gamma^2(t)dt.$$

(ii) If $X(t) = J(t)$, then $f(X(t))$ is also a pure jump process. Moreover,

$$f(X(t)) = f(X(0)) + \sum_{0<s\leq t} f(X(s)) - f(X(s-)).$$

(iii) In general when $X(t) = X^c(t) + J(t)$, intuitively we should have $df(X(t))$ following the classical Ito's formula in between the jumps of X and $\Delta f(X(t)) = f(X(t)) - f(X(t-)$ at the jump points of X.

This leads to the following Ito's formula (see Shreve's theorem 11.5.1)

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))dX^c(s) + \int_0^t \frac{1}{2}f''(X(s))\gamma^2(s)ds \\ &\quad + \sum_{0<s\leq t} f(X(s)) - f(X(s-)). \end{aligned}$$

Remark 10.1. *In general, the above Ito's formula does NOT have a differential form, i.e.* $df(X(t)) = \cdots$*. The reason is generally we cannot express* $\Delta f(X(s)) = f(X(s)) - f(X(s-))$ *in terms of some derivative of* f *and* $\Delta X(s) = X(s) - X(s-)$*. In some special case, for example when* $X(t)$ *is a pure jump process then we may have a differential form for* $df(X(t))$*, but this is not guaranteed. See also the discussion in* (11.2)*.*

10.2 Ito formula for multiple jump processes

Following similar argument to the one dimensional Ito formula for jump process, we can derive the multi-dimensional Ito formula for jump processes. Here we give the version for two processes. The formula for higher dimension follows the same pattern.

Theorem 10.2. *Let X^1, X^2 be two jump processes:*

$$X^i(t) = X^i(0) + \int_0^t \alpha^i(s)ds + \int_0^t \gamma^i(s)dWs + J^i(t), i = 1, 2.$$

Let $f(t, x_1, x_2)$ be a twice differentiable in its spatial variables. Then

$$\begin{aligned}
f(t, X_t^1, X_t^2) &= f(t, X_0^1, X_0^2) + \int_0^t \sum_{i=1}^{2} f_{x_i}(s, X_s^1, X_s^2) d(X^i)_s^c \\
&+ \frac{1}{2} \int_0^t \sum_{i=1}^{2} f_{x_i x_i}(s, X_s^1, X_s^2)(\gamma^i)_s^2 ds \\
&+ \int_0^t f_{x_1 x_2}(s, X_s^1, X_s^2)\gamma_s^1 \gamma_s^2 ds \\
&+ \sum_{0<s\leq t} f(s, X_s^1, X_s^2) - f(s, X_{s-}^1, X_{s-}^2).
\end{aligned}$$

Corollary 10.3. *Let X^1, X^2 be two jump processes:*

$$X^i(t) = X^i(0) + \int_0^t \alpha^i(s)ds + \int_0^t \gamma^i(s)dWs + J^i(t), i = 1, 2.$$

Then

$$\begin{aligned}
X_t^1 X_t^2 &= X_0^1 X_0^2 + \int_0^t X_s^1 d(X^2)_s^c + \int_0^t X_s^2 d(X^1)_s^c \\
&+ \int_0^t \gamma_s^1 \gamma_s^2 ds + \sum_{0<s\leq t} X_s^1 X_s^2 - X_{s-}^1 X_{s-}^2.
\end{aligned}$$

Remark 10.4. *If each X^i is driven by a different Brownian motion W^i and they are independent then the cross variation term in Theorem* (10.2) $\int_0^t f_{x_1x_2}(s, X_s^1, X_s^2)\gamma_s^1\gamma_s^2 ds$ *will disappear, as well as the cross variation term* $\int_0^t \gamma_s^1\gamma_s^2 ds$ *in Corollary* (10.3).

11 Models of stock price with jumps

11.1 Stock models

Recall that before we model the dynamics of a stock $S(t)$ as followed:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dWt.$$

Also observe that the most important property of $W(t)$ we used in pricing financial models with $S(t)$ is that it is a *martingale*. This motivates us to replace $W(t)$ with a general martingale with jumps. That is, we let

$$M(t) = \int_0^t \alpha(s)ds + \int_0^t \gamma(s)dWs + J(t),$$

where $J(t)$ is a pure jump process and $J(t) + \int_0^t \alpha(s)ds$ is a martingale. Consider the following model for $S(t)$:

$$S(t) = S(0) + \int_0^t \mu(s)S(s-)ds + \int_0^t S(s-)dM(s). \tag{14}$$

Intuitively, the reason we use $S(s-)$ in the RHS is so that at the jump of $M(t)$, we have

$$S(t) - S(t-) = S(t-)\Delta J(t). \tag{15}$$

If we think of $\Delta J(t)$ as representing an external shock, then this says the jump in the stock price is its value immediately before the shock occurs multiply with the size of the shock, which makes sense.

Mathematically, using $S(s-)$ in the RHS has the benefit of guaranteeing $\int_0^t S(s-)dM(s)$ to be a martingale under proper conditions (see the discussion in Section 7.2). Either way, it should be noted that we can equivalently write (14) as

$$S(t) = S(0) + \int_0^t (\mu(s) + \alpha(s))S(s)ds + \int_0^t S(s)\gamma(s)dW(s) + \sum_{0<s\leq t} S(s-)\Delta J(s). \tag{16}$$

That is, we only use $S(s-)$ in conjuction with the jumps in $J(s)$.

Relation (15) has another important implication for the jumps of $J(t)$:

$$S(t) = S(t-)(1 + \Delta J(t)).$$

Since we want to use $S(t)$ as a stock price, $S(t) \geq 0$ implies we need to restrict $\Delta J(t) > -1$.

Similar to the classical Black-Scholes model, we have an explicit formula for $S(t)$ satisfying (14) or (16):

$$S(t) = S(0) \exp\Big[\int_0^t [\mu(s) + \alpha(s) - \frac{1}{2}\gamma^2(s)]ds + \int_0^t \gamma(s)dWs\Big] \prod_{0<s\leq t} (1 + \Delta J(s)). \tag{17}$$

Example 11.1. *Geometric Poisson process: If we let* $M(t) = \sigma(N(t) - \lambda t)$ *then*

$$S(t) = S(0) + \int_0^t S(s-)dMs = S(0)e^{-\sigma\lambda t} \prod_{0<s\leq t} (1 + \sigma\Delta N(s)) = S(0)e^{-\sigma\lambda t}(1+\sigma)^{N(t)},$$

since we observe that $1 + \sigma\Delta N(s) = 1 + \sigma$ *at all jump points of* $N(t)$ *and there are exactly* $N(t)$ *jumps at time* t*. Also note that since* σ *is the jump size of the pure jump process* $\sigma N(t)$*, we require* $\sigma > -1$ *as in the discussion above.*

11.2 Some general remarks

Let $W(t)$ be a BM and $N(t)$ be a Poisson process. Observe that

$$X^1(t) = 1 + \int_0^t \sigma X^1(s)ds$$

$$X^2(t) = 1 + \int_0^t \sigma X^2(s)dW(s)$$

$$X^3(t) = 1 + \int_0^t \sigma X^3(s-)dN(s)$$

(note the $X^3(s-)$ in the last equation) have solutions

$$\begin{aligned} X^1(t) &= e^{\sigma t} \\ X^2(t) &= e^{\sigma W(t) - \frac{1}{2}\sigma^2 t} \\ X^3(t) &= (1+\sigma)^{N(t)}, \end{aligned}$$

where the solution for X^1 follows from classical calculus, X^2 from "classical" Ito's formula and X^3 from the calculus for jump processes (see also the discussion about Geometric Poisson process). The point to observe here is that *three very similar differential equations give three distinctly different answers depending on different integrators.*

Also observe that if we apply Ito's formula for jump processes to the $f(N(t)) = (1+\sigma)^{N(t)}$, we get

$$X^3(t) = f(N(t)) = \sum_{s\leq t} (1+\sigma)^{N(s)} - (1+\sigma)^{N(s-)}. \tag{18}$$

This at first glance does not look like the "differential" form

$$\begin{aligned} dX^3(t) &= \sigma X^3(t-)dN(t) \qquad (19) \\ X^3(0) &= 1. \end{aligned}$$

However, we observe from (18) that $\Delta X^3(s) = (1+\sigma)^{N(s)} - (1+\sigma)^{N(s-)}$. Moreover, at the jump point of N

$$\begin{aligned}\Delta X^3(s) &= (1+\sigma)^{N(s)} - (1+\sigma)^{N(s-)} = (1+\sigma)^{N(s)} - (1+\sigma)^{N(s)-1} \\ &= \sigma(1+\sigma)^{N(s)-1} = \sigma X^3(s-) \\ &= \sigma X^3(s-)\Delta N(s).\end{aligned}$$

Now the agreement between (18) and (19) are clear. The point here is that it is not immediate to derive "differential" form from the explicit formula of a jump process. Indeed such differential form is not always possible. The fact that $N(t)$ is a counting process (having jump of size 1) is central to the reason why the formula X^3 is nice, as well as that we could re-derive the differential form of $X^3(t)$ from its explicit formula. Replacing $N(t)$ with a general jump process (having arbitrary jump size) in the differential equation for X^3, and you will see that we no longer can easily derive such nice formula anymore.