Math 622
Spring 2015
Final Exam - Form A
05/12/2015

Name (Print): ____________________________

This exam contains 7 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- The parts of the problems are **not necessarily connected.** If you cannot do one part, you can assume the result of that part, if needed, to do the next part.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. Let X_t have the dynamics

$$\begin{aligned} dX_t &= dt + X_t dW_t \\ X_0 &= x, \end{aligned}$$

where α, σ are given constants, W is a Brownian motion.

Let Y_t have the dynamics

$$\begin{aligned} dY_t &= c_1 Y_t dt + c_2 Y_t dW_t \\ Y_0 &= 1, \end{aligned}$$

where c_1, c_2 are constants to be chosen.

(a) (5 points) Find $d(X_t Y_t)$.

Ans:

$$d(X_t Y_t) = (c_1 X_t Y_t + Y_t + c_2 X_t Y_t)dt + (X_t Y_t + c_2 X_t Y_t)dW_t.$$

(b) (10 points) Observe from the above part that you can choose c_1, c_2 so that the right hand side of $dX_t Y_t$ is free of terms involving X_t. Moreover, we also know the explicit solution for Y_t. By choosing appropriate constants c_1, c_2, solve for an explicit solution of X_t.

Ans: Choosing $c_2 = -1$ and $c_1 = 1$ we see that

$$d(X_t Y_t) = Y_t dt.$$

Therefore

$$X_t Y_t - x = \int_0^t Y_u du,$$

or

$$X_t = (Y_t)^{-1}\Big(x + \int_0^t Y_u du\Big),$$

where

$$\begin{aligned} Y_t &= e^{(c_1 - 1/2 c_2^2)t + c_2 W_t} \\ &= e^{1/2 t - W_t}. \end{aligned}$$

(c) (10 points) Now let X_t have the dynamics

$$\begin{aligned} dX_t &= \alpha dt + \sigma X_{t-} dN_t \\ X_0 &= x, \end{aligned}$$

where N_t is a Poisson process, α, σ are constants. Note that this is not the geometric model that we considered for the stock model, since the dt term is just αdt. Solve for an explit solution of X_t.

Ans: We can do the analysis as followed: If we let $0 < T_1 < T_2 < \cdots$ be the jump times of N_t then it is clear that

$$\begin{aligned} X_t &= x+\alpha t, \;\; 0 \le t < T_1 \\ &= (1+\sigma)(x+\alpha T_1)+\alpha(t-T_1), \;\; T_1 \le t < T_2 \\ &= (1+\sigma)^2(x+\alpha T_1)+\alpha(1+\sigma)(T_2-T_1)+\alpha(t-T_2), \;\; T_2 \le t < T_3, \end{aligned}$$

and so on.

This is a correct answer. An alternative derivation which leads to a closed form formula for X_t is as followed: taking the motivation from the above parts, let c be a constant to be determined, we have

$$d(c^{N_t}X_t) = \alpha c^{N_t}dt + \sum_{s\le t} c^{N_s}X_s - c^{N_{s-}}X_{s-}.$$

Observe that

$$\begin{aligned} c^{N_s}X_s - c^{N_{s-}X_{s-}} &= cc^{N_{s-}}(X_{s-}+\Delta X_s) - c^{N_{s-}}X_{s-} \\ &= c^{N_{s-}}X_{s-}(c+c\sigma-1). \end{aligned}$$

Therefore if we choose $c=\frac{1}{1+\sigma}$ then

$$d\left[\left(\frac{1}{1+\sigma}\right)^{N_t} X_t\right] = \alpha\left(\frac{1}{1+\sigma}\right)^{N_t} dt.$$

It follows that

$$X_t = x(1+\sigma)^{N_t} + \alpha(1+\sigma)^{N_t}\int_0^t \left(\frac{1}{1+\sigma}\right)^{N_s} ds.$$

You should verify that this formula coincides with the other answer we derived above.

2. Let $S(t)$ be the stock price denominated in Euro and $N^f(t)$ be the price of the Euro money market denominated in dollars. Their dynamics under the US domestic risk neutral measure $\tilde{P}$ is given as followed:

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma_1 S(t)d\tilde{W}(t) + S(t-)d(N(t)-\lambda t) \\ dN^f(t) &= rN^f(t)dt + \sigma_2 N^f(t-)d(N(t)-\lambda t), \end{aligned}$$

where under $\tilde{P}$, $\tilde{W}$ is a Brownian motion, $N(t)$ is a Poisson process with rate λ, $N(t)$ and $\tilde{W}$ are independent. r, σ_1, σ_2 are constants, r is the domestic interest rate.

(a) (5 points) Find the explicit solutions for $S(t), N^f(t)$.

Ans: From the lecture note on Chapter 11

$$S(t) = S(0)\exp\left[(r-\lambda-\frac{1}{2}\sigma_1^2)t + \sigma_1\tilde{W}(t) + N(t)\log 2\right]$$
$$N^f(t) = N^f(0)\exp\left[(r-\lambda\sigma_2)t + N(t)\log(1+\sigma_2)\right].$$

(b) (5 points) Define the foreign risk neutral measure as followed

$$\tilde{P}^{N^f}(A) = \tilde{E}\Big(\mathbf{1}_A \frac{D(T)N^f(T)}{N^f(0)}\Big).$$

Recall also the following: if we define a new measure $\tilde{P}^Z$ by

$$\tilde{P}^Z(A) = \tilde{E}\Big(\mathbf{1}_A Z(T)\Big),$$

where

$$Z(T) = \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(T)} e^{-(\tilde{\lambda}-\lambda)T},$$

then $N(t)$ is a Poisson process with rate $\tilde{\lambda}$ under $\tilde{P}^Z$. Use this fact to show that $N(t)$ is a Possion process with rate $\lambda(1+\sigma_2)$ under $\tilde{P}^{N^f}$.

Ans:

$$\begin{aligned} Z(T) = \frac{D(T)N^f(T)}{N^f(0)} &= \exp\Big[-\lambda\sigma_2 T + N(T)\log(1+\sigma_2)\Big] \\ &= \exp\Big[\big\{\lambda - \lambda(1+\sigma_2)\big\}T + N(T)\big\{\log(\lambda(1+\sigma_2)) - \log\lambda\big\}\Big] \\ &= \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(T)} e^{-(\tilde{\lambda}-\lambda)T}, \end{aligned}$$

where $\tilde{\lambda} = \lambda(1+\sigma_2)$. Thus by the change of measure result, $N(t)$ has rae $\tilde{\lambda}$ under $\tilde{P}^{N^f}$.

(c) (5 points) Find the dynamics of $S^{N^f}(t) := \frac{S(t)}{N^f(t)}$ under $\tilde{P}^{N^f}$ and show that it is a martingale under $\tilde{P}^{N^f}$.

Ans:

$$\begin{aligned} S^{N^f}(t) &= \frac{S(0)}{N^f(0)} \exp\left[(-\lambda(1-\sigma_2) - \frac{1}{2}\sigma_1^2)t + \sigma_1 \tilde{W}(t) + N(t)\log\frac{2}{1+\sigma_2}\right] \\ &= S^{N^f}(0)\exp\left[(-\tilde{\lambda}\frac{1-\sigma_2}{1+\sigma_2} - \frac{1}{2}\sigma_1^2)t + \sigma_1\tilde{W}(t) + N(t)\log\left(1 + \frac{1-\sigma_2}{1+\sigma_2}\right)\right]. \end{aligned}$$

Therefore,

$$dS^{N^f}(t) = \frac{1-\sigma_2}{1+\sigma_2} S^{N^f}(t-)d(N(t) - \tilde{\lambda})t + \sigma_1 S^{N^f}(t)d\tilde{W}(t)$$

is a martingale under $\tilde{P}^{N^f}$.

(d) (10 points) Now suppose

$$dN^f(t) = rN^f(t)dt + \sigma_3 N_t^f d\tilde{W}_t + \sigma_2 N^f(t-)d(N(t) - \lambda t).$$

That is we added a Brownian motion component to the dynamics of the foreign money market. Find the dynamics of $S^{N^f}(t) := \frac{S(t)}{N^f(t)}$ under $\tilde{P}^{N^f}$ and show that it is a martingale under $\tilde{P}^{N^f}$.

Ans: We now have

$$\begin{aligned} N^f(t) &= N^f(0)\exp\left[(r - \lambda\sigma_2 - 1/2(\sigma_3)^2)t + \sigma_3\tilde{W}_t + N(t)\log(1+\sigma_2)\right] \\ Z(T) &= \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(T)} e^{-(\tilde{\lambda}-\lambda)T} e^{-\frac{1}{2}(\sigma_3)^2 T + \sigma_3 \tilde{W}_T}, \end{aligned}$$

where $\tilde{\lambda} = \lambda(1+\sigma_2)$. Therefore, under $\tilde{P}^{N^f}$, $N(t)$ is a Possion process with rate $\lambda(1+\sigma_2)$ and $\tilde{W}_t^{N^f} = \tilde{W}_t - \sigma_3 t$ is a Brownian motion.

Moreover,

$$\begin{aligned} S^{N^f}(t) &= \frac{S(0)}{N^f(0)}\exp\left[(-\lambda(1-\sigma_2) - \frac{1}{2}\sigma_1^2)t + (\sigma_1 - \sigma_3)\tilde{W}(t) + 1/2(\sigma_3)^2 t + N(t)\log\frac{2}{1+\sigma_2}\right] \\ &= S^{N^f}(0)\exp\left[(-\tilde{\lambda}\frac{1-\sigma_2}{1+\sigma_2} - \frac{1}{2}(\sigma_1-\sigma_3)^2)t + (\sigma_1-\sigma_3)\tilde{W}^{N_f}(t) + N(t)\log\left(1+\frac{1-\sigma_2}{1+\sigma_2}\right)\right]. \end{aligned}$$

Therefore,

$$dS^{N^f}(t) = \frac{1-\sigma_2}{1+\sigma_2}S^{N^f}(t-)d(N(t) - \tilde{\lambda})t + (\sigma_1 - \sigma_3)S^{N^f}(t)d\tilde{W}^{N_f}(t)$$

and it is a $\tilde{P}^{N_f}$ martingale.

3. Consider the following Hull-White model for interest rate under the risk neutral measure:

$$\begin{aligned} dR_t &= -R_t dt + d\tilde{W}_t \\ R_0 &= 0. \end{aligned}$$

(a) (10 points) Find the explicit solution for R_t.
Ans: Since

$$d(e^t R_t) = e^t d\tilde{W}_t,$$

we have

$$\begin{aligned} R_t &= e^{-t}[R_0 + \int_0^t e^u d\tilde{W}_u] \\ &= e^{-t} \int_0^t e^u d\tilde{W}_u. \end{aligned}$$

(b) (15 points) Find the explicit solution for $B(0,T)$.
Ans: We have

$$\begin{aligned} \int_0^T R_u du &= \int_0^T e^{-u} \int_0^u e^s d\tilde{W}_s du \\ &= \int_0^T e^s \int_s^T e^{-u} du d\tilde{W}_s \\ &= \int_0^T (1 - e^{s-T}) d\tilde{W}_s. \end{aligned}$$

Thus $-\int_0^T R_u du$ has Normal $(0, \sigma_T^2)$ distribution where

$$\sigma_T^2 = \int_0^T (1 - e^{s-T})^2 ds.$$

Thus by the moment generating function formula,

$$B(0,T) = \tilde{E}\left(e^{-\int_0^T R_u du}\right) = e^{\frac{1}{2}\sigma_T^2}.$$

4. Suppose for all T, the zero coupon bond $B(t,T)$ has the following dynamics under the risk neutral measure:

$$dB(t,T) = R(t)B(t,T)dt - \sigma^*(t,T)B(t,T)d\tilde{W}_t,$$

where $\sigma^*(t,T)$ is given for all t, T and $\frac{\partial}{\partial T}\sigma^*(t,T)$ is well-defined.
Recall that the LIBOR rate with tenor δ $L_\delta(t,T)$ is defined as

$$B(t,T) = (1 + \delta L_\delta(t,T))B(t,T+\delta).$$

(a) (10 points) Find the dynamics of $L_\delta(t,T)$ under the risk neutral measure $\tilde{P}$ (Note: NOT under the forward measure $\tilde{P}^{T+\delta}$ as we did in class).

Ans: We have shown, under $\tilde{P}^{T+\delta}$,

$$dL_\delta(t,T) = (1/\delta + L_\delta(t,T))(\sigma^*(t,T+\delta) - \sigma^*(t,T))d\tilde{W}_t^{T+\delta}.$$

Also from the change of numéraire result,

$$d\tilde{W}_t^{T+\delta} = d\tilde{W}_t + \sigma^*(t,T+\delta)dt.$$

Therefore,

$$dL_\delta(t,T) = (1/\delta + L_\delta(t,T))(\sigma^*(t,T+\delta) - \sigma^*(t,T))d(\tilde{W}_t + \sigma^*(t,T+\delta)dt).$$

(b) (15 points) Suppose that $\sigma^*(t,T) = T - t$, for all t, T. Find an explicit solution for $L_\delta(t,T)$ in terms of $L_\delta(0,T), \delta, \tilde{W}_t^{T+\delta}$.

Ans: Observe that

$$\begin{aligned} d(1+\delta L_\delta(t,T)) &= \delta dL_\delta(t,T) = (1+\delta L_\delta(t,T))(\sigma^*(t,T+\delta) - \sigma^*(t,T))d\tilde{W}_t^{T+\delta} \\ &= \delta(1+\delta L_\delta(t,T))d\tilde{W}_t^{T+\delta}. \end{aligned}$$

Therefore,

$$1 + \delta L_\delta(t,T) = \big(1 + \delta L_\delta(0,T)\big)e^{\delta \tilde{W}_t^{T+\delta} - \frac{1}{2}\delta^2 t}.$$

That is

$$L_\delta(t,T) = \frac{1}{\delta}\Big[\big(1 + \delta L_\delta(0,T)\big)e^{\delta \tilde{W}_t^{T+\delta} - \frac{1}{2}\delta^2 t} - 1\Big].$$