Math 485
Fall 2019
Midterm 2
11/14/19

Name (Print): ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

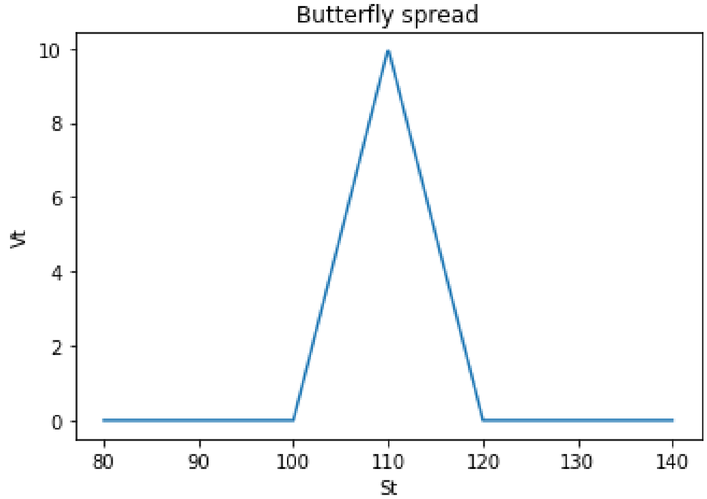
You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the multiperiod binomial model with $r = 0.02, u = 1.03, d = .98$ and $\Delta T = 1$. Suppose $S_0 = 110$. (Note $r \neq 0$ here). Recall that a butterfly spread $100, 110, 120$ has the following pay off function:



Find the risk neutral price V_0 of an **American** option on this spread with expiration $N = 3$.

Ans: $q = \frac{e^{-r\Delta T} - d}{u-d} = 0.80$.

Event	S	$V_{exercise}$	V_{cont}	V	Decision
uuu	120.2	0	N/A	0	N/A
uud	114.36	5.63	N/A	0	N/A
udd	108.81	8.81	N/A	8.81	N/A
ddd	103.53	5.53	N/A	5.53	N/A
uu	116.7	3.30	1.08	3.30	exercise
$ud = du$	111.03	8.96	6.13	8.96	exercise
dd	105.64	5.64	7.62	7.62	cont
u	113.3	6.7	4.32	6.7	exercise
d	107.8	7.8	8.53	8.53	cont
$\varnothing$	110	10	6.9	10	exercise

Thus $V_0 = 10$ for this derivative.

2. (a) (10 points) Consider a forward contract with expiration T and an American option component. That is the holder of the contract at any time t between $(0, T)$, if he/she wishes, can exercise the contract and receives a pay off of $S_t - K$. If the person did not exercise at time $t < T$ he/she will have to exercise at time T. Find the price of this (so called) American forward contract. Assume $r > 0$.

   Ans: The continuation value is at least $S_t - Ke^{-r(T-t)}$ because the holder can always treat it as a European forward and exercise only at T. Since $S_t - Ke^{-r(T-t)} > S_t - K$, the holder would never exercise early. Thus the price of this American forward contract is exactly the same as the European forward contract : $V_t = S_t - Ke^{-r(T-t)}$.

   (b) (10 points) Does the put call parity hold for the American equivalences? That is

$$V_0^{\text{American call}} - V_0^{\text{American put}} = V_0^{\text{American forward}},$$

   where $V_0^{\text{American forward}}$ is the contract we described in part a? Explain.

   Ans: We have discussed that $V_0^{\text{American call}} = V_0^{\text{Euro call}}$ while $V_0^{\text{American put}} > V_0^{\text{Euro put}}$ in general when $r > 0$. From part a, $V_0^{\text{American forward}} = V_0^{\text{Euro forward}}$. Since put call parity holds in the European case, these imply that put call parity does not hold in general in the American case.

3. Let W_t be a Brownian motion. Perform the following calculations:

(a) (5 points) $E(e^{W_t}|W_s), s < t$

Ans:

$$\begin{aligned} E(e^{W_t}|W_s) &= E(e^{W_t - W_s + W_s}|W_s) \\ &= e^{W_s}E(e^{W_t - W_s}|W_s) = e^{W_s}E(e^{W_t - W_s}) \\ &= e^{W_s + \frac{t-s}{2}}. \end{aligned}$$

(b) (5 points) $P(\int_0^t e^s dW_s > t)$

Ans:

$$\begin{aligned} P(\int_0^t e^s dW_s > t) &= P(N(0, \int_0^t e^{2s} ds) > t) \\ &= P(Z > \sqrt{\frac{2}{e^{2t} - 1}} t) = N(-\sqrt{\frac{2}{e^{2t} - 1}} t). \end{aligned}$$

(c) (5 points) $E(\int_0^t e^{W_s} ds)$

Ans:

$$\begin{aligned} E(\int_0^t e^{W_s} ds) &= \int_0^t E(e^{W_s}) ds \\ &= \int_0^t e^{\frac{s}{2}} ds = 2(e^{\frac{t}{2}} - 1). \end{aligned}$$

(d) (5 points) $E(\int_0^t W_s dW_s)^2$

Ans:

$$E(\int_0^t W_s dW_s)^2 = \int_0^t E(W_s^2) ds = \int_0^t s ds = \frac{t^2}{2}.$$

4. Consider the Vasicek model for interest rate:

$$dr_t = k(\theta - r_t)dt + \sigma dW_t.$$

Here k, θ, σ are constants.

(a) (10 points) Find $E(r_t)$ supposing that the initial rate r_0 is given.

Ans:

$$r_t - r_0 = \int_0^t k(\theta - r_s)ds + \int_0^t \sigma dW_s.$$

Thus

$$E(r_t) = r_0 + \int_0^t k(\theta - E(r_s))ds.$$

If we let $f(t) = r_t$ then

$$\begin{aligned} f' &= k(\theta - f) \\ f(0) &= r_0. \end{aligned}$$

Solving using integrating factor gives

$$\begin{aligned} (e^{kt} f)' &= k\theta e^{kt} \\ f &= ce^{-kt} + \theta. \end{aligned}$$

Plug in the initial condition gives $c = r_0 - \theta$. Thus

$$E(r_t) = (r_0 - \theta)e^{-kt} + \theta.$$

(b) (5 points) Find $d(e^{kt} r_t)$

Ans:

$$\begin{aligned} de^{kt} r_t &= ke^{kt} r_t + e^{kt} dr_t \\ &= e^{kt}(kr_t + k(\theta - r_t)dt + \sigma dW_t) \\ &= e^{kt}(k\theta dt + \sigma dW_t). \end{aligned}$$

(c) (5 points) Solve for r_t using the answer in part b.

Ans:

Integrating both sides of part b answer gives

$$\begin{aligned} e^{kt} r_t - r_0 &= \int_0^t e^{ks} k\theta ds + \int_0^t e^{ks} \sigma dW_s \\ &= \theta(e^{kt} - 1) + \sigma \int_0^t e^{ks} dW_s. \end{aligned}$$

Hence we have

$$r_t = e^{-kt} r_0 + \theta(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ks} dW_s.$$

Note that this gives a consistent answer with the one in part a.

5. Consider the Black-Scholes **physical** model:

$$dS_t = \mu S_t dt + \sigma S_t W_t.$$

(a) (10 points) Find the price of a Euro call option on S using the following parameters: $S_0 = 100, K = 100, T = 1, r = 0.02, \mu = 0.07, \sigma = 0.3$ (a numerical answer is expected here).

Ans: We have

$$\begin{aligned} d+ &= 0.216 \\ d- &= -0.08 \\ V_0^{call} &= S_0 N(d+) - Ke^{-rT} N(d-) = 12.82 \end{aligned}$$

(b) (5 points) Find the price of a Euro put option on S using the following parameters: $S_0 = 100, K = 100, T = 1, r = 0.02, \mu = 0.07, \sigma = 0.3$ (a numerical answer is expected here).

Ans: From put call parity:

$$V_0^{put} = V_0^{call} - (S_0 - Ke^{-rT}) = 10.84$$

(c) (5 points) Compute the **phyiscal** probability that the call option will expire **out of the money** (below the strike price) using the following parameters: $S_0 = 100, K = 100, T = 1, r = 0.02, \mu = 0.07, \sigma = 0.3$ (a numerical answer is expected here).

Ans: We have : $d- = \frac{(\mu - \frac{1}{2}\sigma^2)T - \log(K/S_0)}{\sigma\sqrt{T}} = 0.083$. The probability of being out of the money is $N(-d-) = 0.47$.