Math 485
Fall 2019
Midterm exam 1
10/10/19

Name (Print): ______________________________

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

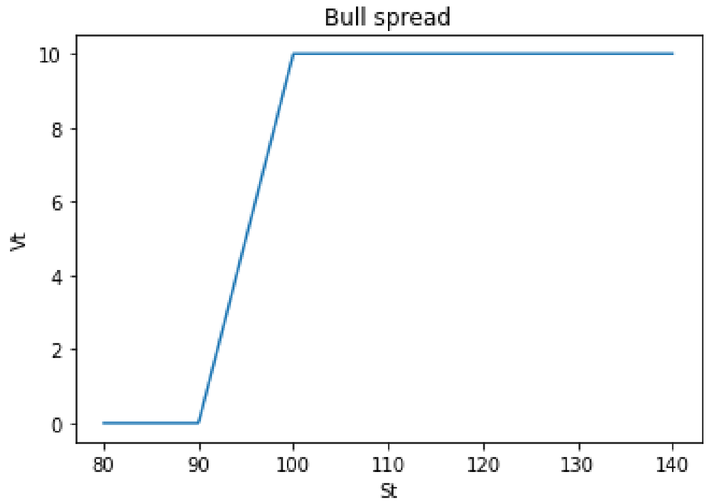
You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

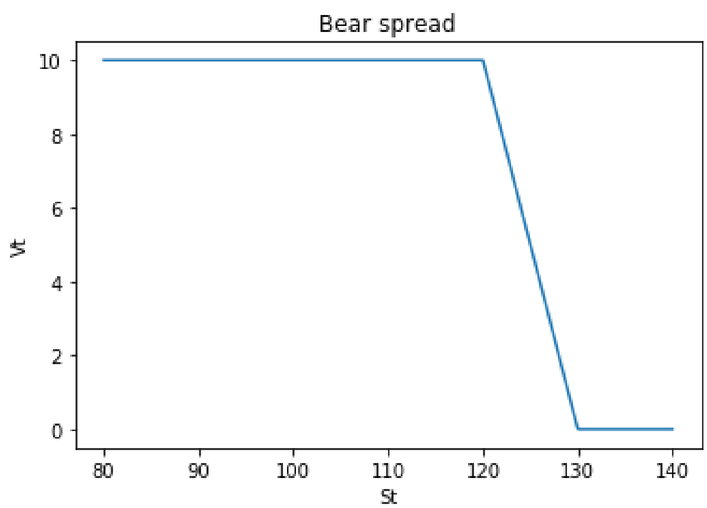
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

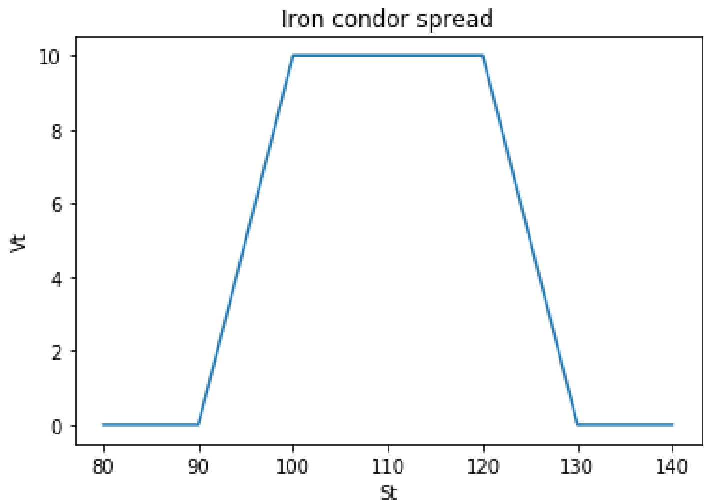
1. (a) (5 points) Recall that a bull spread is created by longing a call option with strike K_1 and shorting a call option with strike $K_2, K_1 < K_2$ on the same stock with the same expiry. That is $V_T^{bull,K_1,K_2} = V_T^{call,K_1} - V_T^{call,K_2}, K_1 < K2$. Plot the payoff function of a bull spread $V_T^{bull,90,100}$.



(b) (5 points) Recall that a bear spread is created by longing a put option with strike K_2 and shorting a put option with strike $K_1, K_1 < K_2$ on the same stock with the same expiry. That is $V_T^{bear,K_1,K_2} = V_T^{put,K_2} - V_T^{put,K_1}, K_1 < K2$. Plot the payoff function of a bear spread $V_T^{bear,120,130}$.

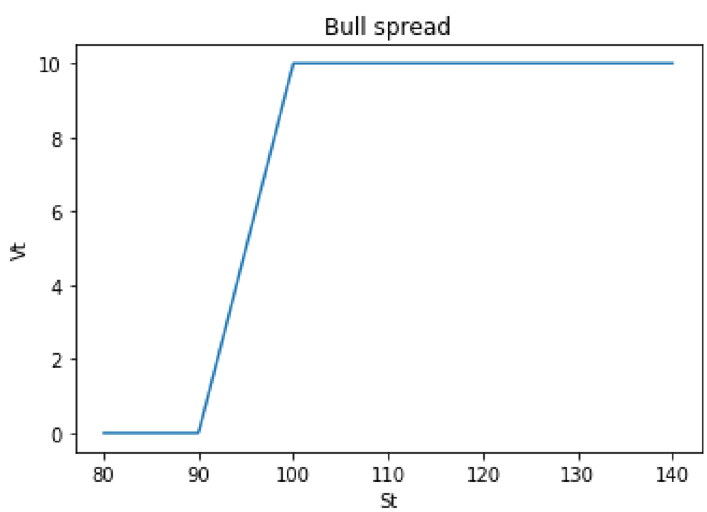


(c) (10 points) An iron condor spread has the following payoff function. Find a replicating portfolio for this iron condor spread using only call and put options.



Ans: It is tempting to replicate an iron codor spread with longing a bull spread and a bear spread. However this is not correct as the two payments overlap. For example, when $100 \le S_T \le 120$, the payments from both bull and bear spreads will give a total payment of 20 dollars if one longs both of those but the iron condor spread only pays 10 dollars.

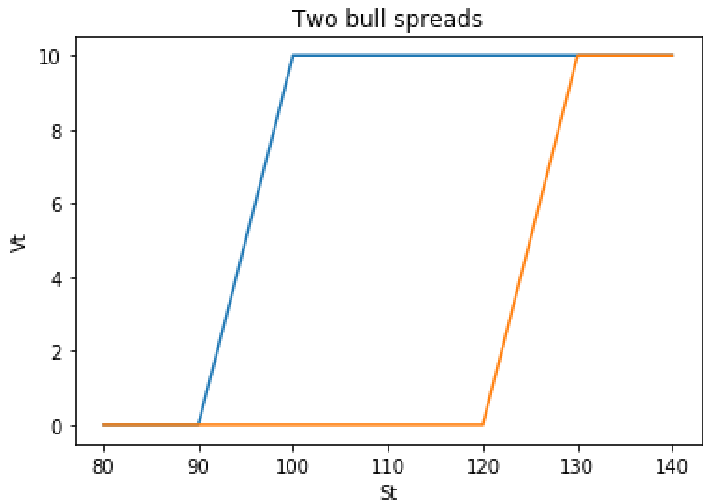
To replicate the iron condor spread then we can start with a 90, 100 bull spread:



To achieve the payoff of the iron condor, we need to "bring down" the payment after $S_T = 120$. "Bringing down" points to shorting something. Since the payment is exactly 0 after $S_T = 130$, this suggests we short 120, 130 bull spread:

In fact, we just need to check that the payment of this portfolio (longing one 90,100 bull spread and shorting one 120, 130 bull spread) matches the iron condor for $S_T \ge 120$.

- For $120 \le S_T \le 130$, the payment is 10 from the 90,100 bull spread and $-(S_T - 120)$ from the short position of the 120,130 bull spread. Thus the total payment is $130 - S_T$, which is exactly the iron condor payment.



- For $130 \leq S_T$, the paymetn is 10 from the 90,100 bull spread and -10 from the short position of the 120,130 bull spread. Thus the total payment is 0, which is also exactly the iron condor payment.

In terms of holding only long and put options, the replicating portfolio then is longing 1 share of 90 call, shorting 1 share of 100 call, shorting one share of 120 call and longing one share of 130 call.

An alternative solution is to start with the 120, 130 bear spread and short a 90,100 bear spread with similar reasoning. This results in a portfolio of longing a 130 put, shorting a 120 put, shorting a 100 put and longing a 90 put. Both replicate the 90,100, 120, 130 iron condor.

2. (a) (10 points) Consider a one period binomial model with $r = 0.02, u = 1.05, d = .9$. Suppose $S_0 = 110$. Find the price V_0 of the iron condor spread on S described in part c of the previous problem with $T = 1$.

   Ans: $q = \frac{e^{r\Delta T} - d}{u-d} = .8$

Path	Probability	Stock value	Option value
u	0.8	115.5	10
d	0.2	99	9.

   Thus $V_0 \approx 9.6$

   (b) (10 points) Consider a 3-period binomial model with $r = 0.01, u = 1.02, d = .98$. Suppose $S_0 = 110$. Find the price V_0 of the iron condor spread on S described in part c of the previous problem with $T = 1$.

   Ans: $q = \frac{e^{r\Delta T} - d}{u-d} = .58$

Path	Probability	Stock value	Option value
uuu	0.2	116.73	10
uud	0.43	112.15	10
udd	0.4	107.75	10
ddd	.07	103.53	10

   Thus $V_0 \approx 9.9$

3. Consider a multiperiod model where $S_0 = 100, u = 1.02, d = 0.98, r = 0.01, \Delta T = 0.5$. Calculate

   (a) (5 points)

$$\tilde{E}(\frac{S_5 - S_3}{S_3}|S_3).$$

Ans: $q = \frac{e^{r\Delta T} - d}{u-d} \approx .63$

$$\begin{aligned}
\tilde{E}(\frac{S_5 - S_3}{S_3}|S_3) &= \tilde{E}(\frac{S_3(X_4X_5 - 1)}{S_3}|S_3) \\
&= \tilde{E}(X_4X_5 - 1) \\
&= (qu + (1-q)d)^2 - 1 \approx 0.01
\end{aligned}$$

   (b) (5 points)

$$\tilde{E}(\ln \frac{S_5}{S_2}|S_2).$$

Ans:

$$\begin{aligned}
\tilde{E}(\ln \frac{S_5}{S_2}|S_2) &= \tilde{E}(\ln \frac{S_2X_3X_4X_5}{S_2}|S_2) \\
&= \tilde{E}\ln X_3X_4X_5 \\
&= 3(q\ln u + (1-q)\ln d) \approx 0.0144
\end{aligned}$$

   (c) (10 points)

$$\tilde{E}(\left(\frac{S_2 - S_1}{S_1}\right)^2|S_1).$$

Ans:

$$\begin{aligned}
\tilde{E}(\left(\frac{S_2 - S_1}{S_1}\right)^2|S_1) &= \tilde{E}(\left(\frac{S_1(X_2 - 1)}{S_1}\right)^2|S_1) \\
&= \tilde{E}((X_2-1)^2) = (qu + (1-q)d)^2 - 2(qu + (1-q)d) + 1 \\
&\approx 2.5 \times 10^{-5}
\end{aligned}$$

4. Mr. Johnny Pricealot has come up with a multi-period model for Oranges Inc stock S for the next 4 years (so $n = 4$ and $\Delta T = 1$) where $S_0 = 100, u = 1.05, d = .98, r = 0.02$. Johnny has great confidence in Oranges Inc performance in the next 4 years so he proposes the following exotic option : the holder will get one share of S if the stock has gone up 2 times or more in the next 4 years, and will get nothing otherwise.

   (a) (10 points) Find the no arbitrage price V_0 Johnny can get for this option now.

   $q = \frac{e^{r\Delta T} - d}{u-d} \approx 0.57$

   Since the option is worthless for any path that has 1 u or less, we only need to sum over the paths with 2 u's or more. That is

$$\begin{aligned} V_0 &= e^{-4r}\Big(\binom{4}{2}q^2(1-q)^2 S_0 u^2 d^2 + \binom{4}{3}q^3(1-q)S_0 u^3 d \\ &+ \binom{4}{4}q^4 S_0 u^4\Big) \\ &\approx 81.03 \end{aligned}$$

   (b) (10 points) Find $V_1(d)$ of this option.

   Here we also only need to sum over the paths with 2 u's or more starting from time 1 and outcome d. That is

$$\begin{aligned} V_1(d) &= e^{-3r}\Big(\binom{3}{2}q^2(1-q)S_0 d u^2 d + \binom{3}{3}q^3 S_0 d u^3 \\ &\approx 62.24 \end{aligned}$$

5. In the following market models, decide if there is an arbitrage opportunity. If no, explain why. If yes, provide an arbitrage portfolio.

   (a) (5 points) There are two states in the future universe : ω_1, ω_2 and two assets. We have

$$\begin{aligned} S_T^1(\omega_1) &= S_0^1 u_1, S_T^1(\omega_2) = S_0^1 d_1, \\ S_T^2(\omega_1) &= S_0^2 u_2, S_T^2(\omega_2) = S_0^2 d_2. \end{aligned}$$

   $u_1 = 1.02, u_2 = 1.05, d_1 = 0.95, d_2 = 0.98, r = 0, T = 1, S_0^1 = 50, S_0^2 = 150.$

   The theme of this problem is to solve for a risk neutral probability. If there exists one then there's no arbitrage. If there doesn't exist one then we try to construct an arbitrage opportunity. Here the risk neutral probability satisfies

$$\begin{aligned} q51 + (1-q)47.5 &= 50 \\ q157.5 + (1-q)147 &- 150. \end{aligned}$$

   Multiplying the first equation by 3 gives

$$\begin{aligned} q153 + (1-q)142.5 &= 150 \\ q157.5 + (1-q)147 &- 150. \end{aligned}$$

   It is easy to see that there is no solution here. An arbitrage opportunity can be constructed by longing 1 share of S^2 and shorting 3 shares of S^1. This can be seen from the fact that the returns of S^2 is always higher than S^1. (This is problem 1 from homework 3)

   (b) (5 points) There are two states in the future universe : ω_1, ω_2 and two assets. We have

$$\begin{aligned} S_T^1(\omega_1) &= S_0^1 u_1, S_T^1(\omega_2) = S_0^1 d_1, \\ S_T^2(\omega_1) &= S_0^2 d_2, S_T^2(\omega_2) = S_0^2 u_2. \end{aligned}$$

   $u_1 = 1.02, u_2 = 1.05, d_1 = 0.95, d_2 = 0.98, r = 0, T = 1, S_0^1 = 100, S_0^2 = 100.$

   Here note that S^1 goes up when S^2 goes down and vice versa. If we let q be the risk neutral probability of ω_1 then it satisfies

$$\begin{aligned} q102 + (1-q)95 &= 100 \\ q98 + (1-q)105 &= 100. \end{aligned}$$

   Here $q = 5/7$ is the unique solution so there is no arbitrage here.

   (c) (10 points) There are two states in the future universe : ω_1, ω_2 and two assets. We have

$$\begin{aligned} S_T^1(\omega_1) &= S_0^1 u_1, S_T^1(\omega_2) = S_0^1 d_1, \\ S_T^2(\omega_1) &= S_0^2 u_2, S_T^2(\omega_2) = S_0^2 d_2. \end{aligned}$$

   $u_1 = u_2 = 1.05, d_1 = 0.95 = d_2 = 0.95, r = 0, T = 1, S_0^1 = S_0^2 = 100.$ S^1 pays dividend with rate $\delta = 0.01$.

   This is problem 1 and 2 in homework 4. Since S^1 pays dividend,it is more desirable to hold S^1. An arbitrage portfolio is to long 1 share of S^1 and short 1 share of S^2. This costs 0 dollar to form and at time 1, we have $e^{r\delta}(>1)$ shares of S^1 and only need to payback 1 share of S^2. Thus this is an arbitrage.