Math 485
Fall 2019
Final exam
12/17/19

Name (Print): ____________________

This exam contains 9 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

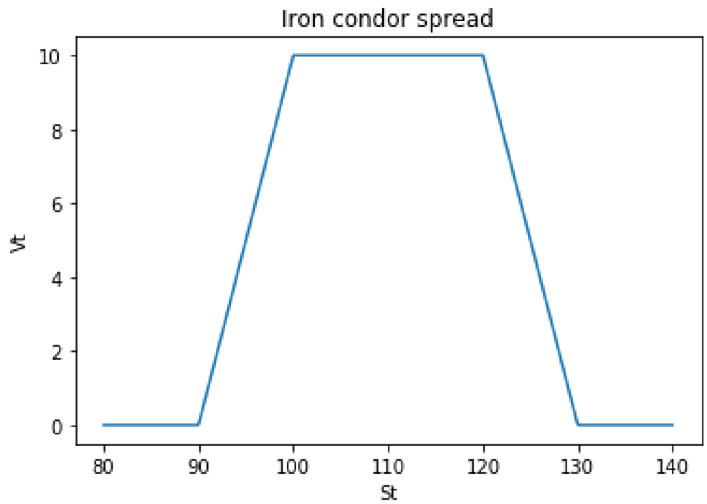
You may use 1 pages of note (two sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	35	
2	15	
3	15	
4	25	
5	20	
6	25	
Total:	135	

1. Consider the multiperiod binomial model with $r = 0.02, u = 1.03, d = .98$ and $\Delta T = 1$. Suppose $S_0 = 110$. (Note $r \neq 0$ here). Recall that a iron condor spread 90, 110, 130 has the following pay off function:



(a) (5 points) Find the risk neutral price V_0 of a **European** option on this spread with expiration $N = 3$.

Ans: $q = \frac{e^{r\Delta T} - d}{u - d} = 0.804$

Path	Stock value	Spread value	Prob
uuu	120.2	9.8	q^3
uud	114.37	10	$3q^2(1-q)$
udd	108.81	10	$3q(1-q)^2$
ddd	103.53	10	$(1-q)^3$

Thus $V_0 \approx 9.32$

(b) (5 points) Find the risk neutral price V_0 of an **American** option on this spread with expiration $N = 3$.

Ans: At time 0, the option holder can choose either to exercise and receive 10 dollars or continue. The continuation value cannot be higher than 10 dollars because of the shape of the payoff function (10 is the max value). This together with $r > 0$ implies that optimal strategy is to exercise at time 0. Thus $V_0 = 10$.

(c) (5 points) Prove that the following portfolio : longing 1 share of 90 call, shorting 1 share of 100 call, shorting one share of 120 call and longing one share of 130 call replicates the iron condor spread of this problem.

Ans: We only need to check that the two portfolio values agree at time T.

- $S < 90$: all call options are out of the money so $V_T^{\text{portfolio}} = 0 = V_T^{\text{spread}}$.
- $90 \leq S < 100$: the only call option in the money is the 90 call. This one is worth $S - 90$ in this case, which agrees with V_T^{spread}.
- $100 \leq S < 120$: Two options are in the money: 90 call and 100 call. Together they are worth $S - 90 - (S - 100) = 10$, which agrees with V_T^{spread}.

- $120 \leq S < 130$: All options except the 130 call are in the money. Together they are worth $S - 90 - (S - 100) - (S - 120) = 130 - S$, which again agrees with V_T^{spread}.
- $130 \leq S$: All options are in the money. Together they are worth $S - 90 - (S - 100) - (S - 120) + (S - 130) = 0 = V_T^{\text{spread}}$.

(d) (5 points) From here on assume a Black-Scholes model with $r = 0.02, \mu = 0.07, \sigma = 0.3$ and $T = 1$. Suppose $S_0 = 110$. Find the vega of the iron condor spread. Does the sign of the vega agree with your intuition ? Explain.

$$\begin{aligned}
\nu^{\text{call, 90}} &= S_0\sqrt{T}\phi_Z(d_{+,90}) \approx 110 \times 0.2695 \\
\nu^{\text{call, 100}} &= S_0\sqrt{T}\phi_Z(d_{+,100}) \approx 110 \times 0.3459 \\
\nu^{\text{call, 120}} &= S_0\sqrt{T}\phi_Z(d_{+,120}) \approx 110 \times 0.3979 \\
\nu^{\text{call, 130}} &= S_0\sqrt{T}\phi_Z(d_{+,130}) \approx 110 \times 0.3765.
\end{aligned}$$

Thus $\nu^{\text{iron spread}} = \nu^{\text{call, 90}} - \nu^{\text{call, 100}} - \nu^{\text{call, 120}} + \nu^{\text{call, 130}} = -10.88$. This agrees with intuition as the volatility increases the spread has more chance to be out of the money.

(e) (5 points) Find the **physical** probability that the spread expires in the money using the information from the previous part.

Ans:

$$\begin{aligned}
P(90 < S_T < 130) &= P(S_T < 130) - P(S_T < 90) \\
&= 1 - N(d_{-,130} - N(d - .90) = 0.4542,
\end{aligned}$$

where here we mean

$$N(d_{-,K}) = \frac{(\mu - \frac{1}{2}\sigma^2)T - \ln K/S_0}{\sigma\sqrt{T}}.$$

(f) (5 points) A portfolio consists of 1 share of this iron condor. Find the number of shares of S to add to the portfolio make it Delta neutral.

Ans:

$$\begin{aligned}
\Delta^{\text{call, 90}} &= N(d_{+,90}) \approx 0.8133 \\
\Delta^{\text{call, 100}} &= N(d_{+,100}) \approx 0.7019 \\
\Delta^{\text{call, 120}} &= N(d_{+,120}) \approx 1 - 0.5279 \\
\Delta^{\text{call, 130}} &= N(d_{+,130}) \approx 1 - 0.6331.
\end{aligned}$$

Thus $\Delta^{\text{iron spread}} = \Delta^{\text{call, 90}} - \Delta^{\text{call, 100}} - \Delta^{\text{call, 120}} + \Delta^{\text{call, 130}} = 0.0062$. Since $\Delta S = 1$, we need to add - 0.0062 (short 0.0062) shares of stock to make the portfolio Delta neutral.

(g) (5 points) Find the number of shares of put at strike $K = 110$ and stocks S to add to the portfolio in the previous part to make it Delta and Gamma neutral.

Ans:

$$
\begin{aligned}
\Gamma^{\text{call, } 90} &= \frac{\phi_Z(d_{+,90})}{S_0\sigma\sqrt{T}} \approx \frac{0.2695}{33} \\
\Gamma^{\text{call, } 100} &= \frac{\phi_Z(d_{+,100})}{S_0\sigma\sqrt{T}} \approx \frac{0.3459}{33} \\
\Gamma^{\text{call, } 120} &= \frac{\phi_Z(d_{+,120})}{S_0\sigma\sqrt{T}} \approx \frac{0.3979}{33} \\
\Gamma^{\text{call, } 130} &= \frac{\phi_Z(d_{+,130})}{S_0\sigma\sqrt{T}} \approx \frac{0.3765}{33}.
\end{aligned}
$$

Thus $\Gamma^{\text{iron spread}} = \Gamma^{\text{call, } 90} - \Gamma^{\text{call, } 100} - \Gamma^{\text{call, } 120} + \Gamma^{\text{call, } 130} = -0.0030$. Since $\Gamma^{\text{put, } 110} = \Gamma^{\text{call, } 110} = 0.0118$, we need to add $\frac{0.0030}{0.0118} = 0.2540$ shares of put to make the portfolio Gamma neutral.

The new Delta of this portfolio is $0.0062 + 0.2540\Delta^{\text{put, } 110} = -0.099$. Again since $\Delta S = 1$ we need to add 0.099 shares of stock to finally make the portfolio Delta and Gamma neutral.

2. Consider a multiperiod model where $S_0 = 100, u = 1.02, d = 0.98, r = 0.01, \Delta T = 1$. Calculate

   (a) (5 points) The price of Asian option with expiry $N = 2$.

   Ans: $q = \frac{e^{r\Delta T} - d}{u-d} = 0.75$

Path	Option value	Prob
uu	102.013	q^2
uu	100.653	$q(1-q)$
ud	99.32	$q(1-q)$
du	98.013	$(1-q)^2$

   Thus $V\text{Asian}_0 \approx 99.0082$.

   (b) (5 points) The price of look back option with expiry $N = 2$.

Path	Option value	Prob
uu	104.04	q^2
uu	102	$q(1-q)$
ud	100	$q(1-q)$
du	100	$(1-q)^2$

   Thus $V\text{look back}_0 \approx 100.6212$.

   (c) (5 points) The price of down and out put option with $K = 105, L = 99$ and expiry $N = 2$.

Path	Option value	Prob
uu	0.96	q^2
uu	5.04	$q(1-q)$
ud	0	$q(1-q)$
du	0	$(1-q)^2$

   Thus $V_0^{\text{down and out put}} \approx 1.4542$.

3. In the following market model, decides if there is an arbitrage opportunity. If no, explains why. If yes, provides an arbitrage portfolio.

   (a) (5 points) There are two states in the future universe : ω_1, ω_2 and two assets. We have

   $$\begin{aligned} S_T^1(\omega_1) &= S_0^1 u_1, S_T^1(\omega_2) = S_0^1 d_1, \\ S_T^2(\omega_1) &= S_0^2 u_2, S_T^2(\omega_2) = S_0^2 d_2. \end{aligned}$$

   $u_1 = 1.01, u_2 = 1.04, d_1 = 0.97, d_2 = 0.98, r = 0, T = 1, S_0^1 = 50, S_0^2 = 150.$

   Ans: An arbitrage portfolio is to short 3 shares of S^1 and long 1 share of S^2. Then $V_0 = -3 \times 50 + 150 = 0$. We also have $V_T(\omega_1) = -3 \times 50.5 + 156 = 4.5 > 0$ and $V_T(\omega_2) = -3 \times 48.5 + 147 = 1.5 > 0$.

   (b) (5 points) There are two states in the future universe : ω_1, ω_2 and 1 asset.

   $$S_T(\omega_1) = S_0 u, S_T(\omega_2) = S_0 d$$

   $u = 1.02, d = 0.98, r = 0.03, T = 1, S_0 = 100.$

   Ans: An aribtrage opportunity is to short one share of S_0 and invest into the money market. Then $V_0 = -100 + 100 = 0$. We also have $V_T(\omega_1) = -102 + 100e^{0.03} = 1 > 0, V_T(\omega_2) = -98 + 100e^{0.03} = 5 > 0$.

   (c) (5 points) There are 1 asset S, a call on S and a put on S both with expiry $T = 1$ and strike $K = 90$. The price of the call is $V_0^{\text{call}} = 15$, the put is $V_0^{\text{put}} = 5$ and $S_0 = 100, r = 0$.

   Ans: Here put call parity holds so there is no arbitrage opportunity.

4. Let W_t be a Brownian motion. Perform the following calculations:

   (a) (5 points) $E(e^{tW_t}|W_s), s < t$

   Ans: $E(e^{tW_t}|W_s) = e^{tW_s}E(e^{t(W_t-W_s)}|W_s) = e^{tW_s}e^{\frac{t^2(t-s)}{2}}$.

   (b) (5 points) $P(\int_0^t sdW_s > t)$

   Ans: $\int_0^t sdW_s$ has a $N(0, \frac{t^3}{3})$ distribution. Thus

$$
\begin{aligned}
P(\int_0^t sdW_s > t) &= P(Z > \frac{t}{\sqrt{\frac{t^3}{3}}}) \\
&= 1 - N(\sqrt{\frac{3}{t}}).
\end{aligned}
$$

   (c) (5 points) $E(\int_0^t W_s^2 ds)$

   Ans: $E(\int_0^t W_s^2 ds) = \int_0^t sds = \frac{t^2}{2}$.

   (d) (10 points) $E(\int_0^t e^{W_s}dW_s)^2$

   Ans: $E(\int_0^t e^{W_s}dW_s)^2 = \int E(e^{2W_s}ds)ds = \int_0^t e^{2s}ds = \frac{1}{2}(e^{2t} - 1)$.

5. Consider the Ho-Lee discrete interest rate model :

$$r_k(\omega) = a_k + b_k \times U_k(\omega),$$

where $U_k(\omega)$ is a random variable that counts the number of u's in ω up to and including time k. We use

$$
\begin{aligned}
a_0 &= 0.05 \\
a_1 &= 0.045, b_1 = 0.01 \\
a_2 &= 0.04, b_2 = 0.01.
\end{aligned}
$$

At each step, take the (risk neutral) probability of up to be 1/3 and down to be 2/3. Also take $\Delta T = 1$.

   (a) (5 points) Provide the sketch of the interest rate tree.

   Ans:

Time	Path	r_k
0	NA	0.05
1	u	0.055
1	d	0.045
2	uu	0.06
2	$ud = du$	0.05
2	dd	0.04.

   (b) (5 points) Find the price of the (zero-coupon) bond $B(0, 3)$.

   Ans:

$$B(0,3) = \widetilde{E}\left(\frac{1}{(1+r_0\Delta T)(1+r_1\Delta T)(1+r_2\Delta T)}\right)$$

where

Path	Value	Prob
uu	$\frac{1}{1.05\times1.055\times1.06}$	$(1/3)^2$
ud	$\frac{1}{1.05\times1.055\times1.05}$	$(1/3)(2/3)$
du	$\frac{1}{1.05\times1.045\times1.06}$	$(1/3)(2/3)$
dd	$\frac{1}{1.05\times1.045\times1.04}$	$(1/3)^2$

Add all these results gives $B(0,3) = 0.8653$.

(c) (5 points) Find the price of the interest rate caplet with maturity $N = 3$ and strike $K = 0.05$

Ans:

$$Caplet(0,3) = \widetilde{E}\left(\frac{1}{(1+r_0\Delta T)(1+r_1\Delta T)(1+r_2\Delta T)}(r_2 - K)^+\right)$$

where

Path	Value	Prob
uu	$\frac{1}{1.05\times1.055\times1.06}(.06-.05)^+$	$(1/3)^2$
ud	$\frac{1}{1.05\times1.055\times1.05}(.05-.05)^+$	$(1/3)(2/3)$
du	$\frac{1}{1.05\times1.045\times1.06}(.05-.05)^+$	$(1/3)(2/3)$
dd	$\frac{1}{1.05\times1.045\times1.04}(.04-.05)^+$	$(1/3)^2$

Thus Caplet(0,3) = 0.00095.

(d) (5 points) Find the price of the interest rate cap with maturity $N = 3$ and strike $K = 0.05$.

Ans:

$$\begin{aligned} Caplet(0,1) &= \widetilde{E}\left(\frac{1}{1+r_0\Delta T}(r_0-K)^+\right) = 0 \\ Caplet(0,2) &= \widetilde{E}\left(\frac{1}{(1+r_0\Delta T)(1+r_1\Delta T)}(r_1-K)^+\right) \\ &= \frac{.055-.05}{(1.05\times1.055} \times \frac{1}{3} = 0.0015. \end{aligned}$$

Thus $Cap(0,3) = \sum_{i=1}^{3} Caplet(0,i) = 0.0024$.

6. Consider the Vasicek model for interest rate:

$$dr_t = k(\theta - r_t)dt + \sigma dW_t.$$

Here k, θ, σ are constants.

(a) (5 points) Find $d(e^{kt}r_t)$

Ans:

$$\begin{aligned} de^{kt}r_t &= ke^{kt}r_t + e^{kt}dr_t \\ &= e^{kt}(kr_t + k(\theta - r_t)dt + \sigma dW_t) \\ &= e^{kt}(k\theta dt + \sigma dW_t). \end{aligned}$$

(b) (5 points) Solve for r_t using the answer in part a.

Ans:

Integrating both sides of part b answer gives

$$\begin{aligned} e^{kt}r_t - r_0 &= \int_0^t e^{ks}k\theta ds + \int_0^t e^{ks}\sigma dW_s \\ &= \theta(e^{kt} - 1) + \sigma \int_0^t e^{ks}dW_s. \end{aligned}$$

Hence we have

$$r_t = e^{-kt}r_0 + \theta(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ks}dW_s.$$

(c) (5 points) Find $E(r_t)$ supposing that the initial rate r_0 is given.

Ans: $E(r_t) = e^{-kt}r_0 + \theta(1 - e^{-kt})$.

(d) (5 points) Find $Var(r_t)$ supposing that the initial rate r_0 is given.

From part b

$$r_t = e^{-kt}r_0 + \theta(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ks}dW_s.$$

Since

$$\begin{aligned} \text{Var}(\int_0^t e^{ks}dW_s) &= \int_0^t e^{2ks}ds = \frac{e^{2kt} - 1}{2k}, \\ \text{Var}(r_t) &= \text{Var}(\sigma e^{-kt} \int_0^t e^{ks}dW_s) = \sigma^2 \frac{1 - e^{-2kt}}{2k}. \end{aligned}$$

(e) (5 points) Suppose $r_0 = 0.02, k = 1, \theta = 0.03, \sigma = 0.3$. Find $P(r_1 > 0.03)$ (a numerical answer is expected here).

Ans: We see that r_1 has a N(0.026, 0.0389) distribution. Thus $P(r_1 > 0.03) = P(Z > 0.02) \approx 0.492$.