Math 485
Fall 2016
Midterm exam 2
11/17/16

Name (Print): ______________________________

This exam contains 4 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	15	
3	20	
4	15	
5	30	
Total:	100	

1. (20 points) Derive Black-Scholes formula.

   Ans: See class notes.

2. (15 points) Consider the following model for Google stock under the physical measure:

$$\begin{aligned} dS_t &= (0.05)S_t dt + (0.01)S_t dW_t, \\ S_0 &= 400. \end{aligned}$$

   Find the Call Option price on Google stock with expiry 1 year, interest rate 2% per annum, strike 450 dollars.

   Ans: $d+ \approx -9.773, d- \approx -9.783, V_0 \approx 0$.

3. Consider the following model for interest rate (Hull-White model):

$$dr_t = k(\mu - r_t)dt + \sigma dW_t,$$

   where k, μ, σ are given constant parameters. Suppose r_0 is given.

   (a) (10 points) Find $d(e^{kt} r_t)$.
   Ans:

$$\begin{aligned} d(e^{kt} r_t) &= k e^{kt} r_t dt + e^{kt} dr_t \\ &= e^{kt}(k\mu dt + \sigma dW_t). \end{aligned}$$

   (b) (10 points) Use the result in part a to solve for an explicit formula of r_t. (Your answer may involve expression like $\int_0^t e^s dW_s$ which is fine).
   Ans:

$$\begin{aligned} e^{kt} r_t &= r_0 + k\mu \int_0^t e^{ks} ds + \int_0^t e^{ks} dW_s \\ &= r_0 + \mu(e^{kt} - 1) + \int_0^t e^{ks} dW_s. \end{aligned}$$

   Thus $r_t = e^{-kt} r_0 + \mu(1 - e^{-kt}) + \int_0^t e^{k(s-t)} dW_s$.

4. In the following, let W_t be a Brownian motion. Find

   (a) (5 points) $d(W_t^2 e^t)$
   Ans:

$$d(W_t^2 e^t) = 2W_t e^t dW_t + (W_t^2 e^t + e^t)dt.$$

   (b) (5 points) $d\sin(tW_t)$ Ans:

$$d\sin(tW_t) = [W_t \cos(tW_t) - \frac{1}{2}t^2 \sin(tW_t)]dt + t\cos(tW_t)dW_t.$$

   (c) (5 points) $d(t\cos W_t)$
   Ans:

$$d(t\cos W_t) = (\cos W_t - \frac{1}{2}t\cos(W_t))dt - t\sin(W_t)dW_t.$$

5. (30 points) John Bloomberg is back again with another offer for Mr. T. Right now, Apple stock is 100 per share and Google is 400 per share. This time, Mr. T has 4 call options with strike 100 on Apple each and John has a call option with strike 400 on Google, all with the same expiry in 1 year. From a very trust worthy source, John knows that both Google S_t^G and Apple S_t^A stocks follow Black-Scholes model

$$\begin{aligned} dS_t^G &= (0.05)S_t^G dt + (0.01)S_t^G dW_t, \\ S_0^G &= 400 \\ dS_t^A &= (0.02)S_t^A dt + (0.01)S_t^A dW_t, \\ S_0^A &= 100. \end{aligned}$$

John argues that Mr. T's payoff is $V_T^A = 4(S_T^A - 100)^+ = (4S_T^A - 400)^+$ and John's payoff is $V_T^G = (S_T^G - 400)^+$. But since Google stock grows at a faster rate (0.05 verus 0.02) he offers to exchange his call option for Mr T.'s 4 call options for a small fee of 5 dollars. Should Mr. T agree to this transaction? Why or why not?

Ans: The transaction should not be undertaken, as the value of the two positions are the same. The reason is the call option price must be evaluated under the risk neutral probability, which makes he drift of both stocks r. Since they have the same volatility, the price of 1 call option on S^G with strike 400 is equal to the price of 4 call options on S^A with strike 100.

Scratch (Won't be graded)