Math 485
Fall 2016
Midterm exam 1
10/13/16

Name (Print): ____________________

This exam contains 5 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	20	
3	15	
4	15	
5	15	
6	20	
Total:	100	

1. (15 points) Consider the following model for Google stock : $S_0 = 400$. $S_1 = 500$ with probability 2/3 and $S_1 = 300$ with probability 1/3. Suppose the interest rate is 0. Find the price of a call option on Google stock with strike $K = 250$ and expiration $T = 1$.

   Ans: The risk neutral probability is such that

   $$400 = q500 + (1 - q)300.$$

   Thus q $= 1/2$ and $V_0 = 1/2(250 + 50) = 150$.

2. (20 points) Consider a market with 2 risky assets S^1, S^2 and the interest rate is 0. We have $S_0^1 = 2, S_0^2 = 5$. There are 3 outcomes for S^1, S^2 at time T, denoted by $\omega_1, \omega_2, \omega_3$. Suppose that $S_T^1(\omega_1) = 3, S_T^1(\omega_2) = 1, S_T^1(\omega_3) = 2$ and $S_T^2(\omega_1) = 6, S_T^2(\omega_2) = 1, S_T^3(\omega_3) = 2$. Is the market abitrage free? If yes, explain why. If no, produce an arbitrage portfolio.

   Ans: The risk neutral equation, if exists, must satisfy

   $$\begin{aligned} 2 &= 3q_1 + q_2 + 2(1 - q_1 - q_2) \\ 5 &= 6q_1 + q_2 + 2(1 - q_1 - q_2). \end{aligned}$$

   This equation only has solution $q_1 = 1, q_2 = 0$. Thus the market is not arbitrage free because an equivalent risk neutral probability does not exist. An arbitrage portfolio would be to long 5 shares of S^1 and short 2 shares of S^2. This portfolio has 0 cost at time $t = 0$ and has positive value at all events $\omega_1, \omega_2, \omega_3$ at time $t = 1$.

3. (a) (5 points) Recall that for a real number x, $x^+ = \max(x, 0)$ and $x^- = max(-x, 0)$. Show that for any real number x, $x^+ - x^- = x$.

   Ans: If $x \geq 0$ then $x^+ = x$ and $x^- = 0$. So the relation is true. If $x < 0$ then $x^+ = 0$ and $x^- = -x$ so the relation is also true.

   (b) (10 points) The instructor of Math 485, Mr. T, also trades options on the side to earn extra income. Mr. T has a forward contract and a European put option on Apple stock, expiring on Dec 31st, 2016 both with strike 110 dollars. John Bloomberg, a math 485 student, has a European call option on Apple stock, also expiring on Dec 31st, 2016 with strike 110 dollars. John learns from a very trust-worthy source that Apple stock price has a 60 % chance to rise above 110 and 40 % change to fall below 110 at the end of 2016. So he tries to convince Mr. T to exchange his European call option with Mr. T's put option and forward contract for a small fee of 5 dollars. Should Mr. T agree to this exchange? Why or why not?

Ans: At time T, the value of the call option is $V_T^c = (S_T - K)^+$. The value of the call option is $V_T^p = (S_T - K)^-$. The value of the forward contract is $V_T^f = S_T - K$. From part a :

$$V_T^p = V_T^c + V_T^f.$$

Thus the exchange should cost nothing. So Mr. T should not agree to this exchange.

4. (15 points) Consider a multiperiod binomial model for S with $N = 5, u = 2, d = 1/2, S_0 = 1, r = 0$. Let V be a lookback option on S with expiration $N = 5$. Find $V_3(uuu)$. Explicit number is not necessary. You can leave your answer as an expression of summation.

   Ans: The risk neutral probability is $q = \frac{1-d}{u-d} = 1/3$.

$$V_3(uuu) = 32(1/3)^2 + 16(1/3)(2/3) + 2 \times 8(2/3)^2.$$

5. (15 points) Consider a multiperiod binomial model for S with $N = 5, u = 2, d = 1/2, S_0 = 32, r = 0$. Let V be a down and out put option on S with expiration $N = 5$, strike $K = 30$ and barrier $L = 15$. Find $V_3(ddu)$. Explicit number is not necessary. You can leave your answer as an expression of summation.

   Ans: Since $S_2(dd) = 8 < L = 15$ at the event ddu the option is already striked out. That is $V_3(ddu) = 0$.

6. (20 points) Consider a game where we roll a fair die. If it's odd we receive 1 dollar and if it's 2 or 4 we receive nothing. In all of these events we have the option to roll again. If it's 6 we lose all our winning and the game also ends. The winning is cumulative. What is the risk neutral price for playing this game? What is the optimal strategy?

   Ans: Let $V(n)$ be the value of the game when the winning is n. At the stopping state n we must have

$$n = V(n) > (1/2)(n+1) + (1/3)n.$$

That is $n > 3$. Since we only win 1 dollar at most in each turn, this means we stop when $n = 4$. It remains to decide the probability that we proceed from state n to state $n+1$ before we go bankrupt (hit a 6), for $n \leq 3$. Let this event be E. Conditioning on the outcome of the first throw after reaching state n, we see that

$$P(E) = 1/2 + 1/3P(E).$$

This is because if the first throw is odd (with probability 1/2) then we reach sate $n+1$. If it is 2 or 4 then we go back to state n and start again independently of what happend before. If it's a 6 we hit bankruptcy and has no chance to reach to state $n+1$. Solving this equation gives $P(E) = 3/4$. Since this is true for all state n we have $V(4-k) = 4(3/4)^k, k = 0, 1, 2, 3, 4$. Thus the value of the game is $V(0) = 4(3/4)^4$.

Scratch (Won't be graded)

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