Math 485
Fall 2016
Final exam
12/20/16

Name (Print): ____________________

This exam contains 5 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (two sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	30	
9	30	
Total:	200	

1. (20 points) Derive the Black-Scholes PDE using the game-theory portfolio approach.

   Ans: See class notes.

2. (20 points) Consider a synthetic financial product V where upon an expiration time $T > 0$ the holder of V receives 1 share of a stock S and K dollars in cash. Recall that the forward price for V is an amount $F(0,T)$ that is agreed upon at time 0 such that at time T one can exchange $F(0,T)$ for V free of charge. Suppose S_0 and $B(0,T)$ are known. Express $F(0,T)$ in terms of $K, S_0, B(0,T)$. (Note: the short rate r is NOT assumed to be constant or known in this problem).

   Ans: The present value V_0 of V is $V_0 = S_0 + B(0,T)K$. Thus $F(0,T) = V_0 = S_0 + B(0,T)K$.

3. (20 points) John Bloomberg is back for the last time with an offer for Mr. T ! Mr. T is currently holding a call option on Google with expiration in 1 year. From a very trustworthy source, John learned that the FED will hike the short rate (r) to double its current value tomorrow. John offers to buy out Mr. T's call option at market price (which is determined via Black-Scholes formula upon a Black-Scholes model on Google stock) so that Mr. T does not have to worry about any potential impact of such short rate hike. This time out of generosity John offers his service for free. Assume that besides the short rate, all other market parameters remain contant overnight, as well as the overnight discounting effect is negligible. Should Mr.T agree to this offer? Why or why not?

   Ans: $\frac{\partial V_0}{\partial r} = KTe^{-rT}N(d+) > 0$. Thus as interest rate increases the call option value will increase. Thus selling the call option before the interest rate hike is not recommended.

4. (20 points) Consider a multiperiod binomial model for S with $N = 5, u = 2, d = 1/2, S_0 = 32, r = 0$. Let V be a down and in call option on S with expiration $N = 5$, strike $K = 20$ and barrier $L = 15$. Find $V_3(ddu)$. Explicit number is not necessary. You can leave your answer as an expression of summation.

   Ans: $V_5(dduuu)44$. Otherwise $V_5(ddu*) = 0$ where $* = ud, du, dd$. Thus $V_3(ddu) = 44/9$.

5. (20 points) Consider a 1 period trinomial model where there are 3 possible outcomes at time $T = 1$ (year): $\omega_1, \omega_2, \omega_3$. Suppose that $S_0 = 8, S_T(\omega_1) = 16, S_T(\omega_2) = 6, S_T(\omega_3) = 4$. We also model the spot rate per annum as followed: $r_0 = 0.02, r_T(\omega_1) = 0.01, r_T(\omega_2) = 0.015, r_T(\omega_3) = 0.03$. Interest is compounded discretely once a year, that is 1 dollar at the beginning of year k becomes $1 + r_k$ at the beginning of the year $k + 1$, where r is the spot rate available for the year k. Is this model arbitrage free? Is this model complete? Explain.

   Ans: The risk neutral probabilities $q(\omega_1), q(\omega_2), q(\omega_3)$ have to satisfy

$$\begin{aligned} 8(1+0.02) &= q(\omega_1)8 + q(\omega_2)6q(\omega_3)4 \\ 1 &= q(\omega_1) + q(\omega_2) + q(\omega_3). \end{aligned}$$

   (The interest rate tree is irrelevant in this consideration, except for the value r_0). The RHS matrix is under-determined, and there exists infinitely many solutions. Thus the market is arbitrage free but not complete.

6. (20 points) Let W_t be a Brownian motion. For $s < t$, compute

$$E(\sin(W_t)|W_s).$$

(Hint: Use Euler's formula: $e^{ix} = \cos x + i \sin x$ and the Normal random variable moment generating function).

Ans: $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$. Thus

$$E(\sin(W_t)|W_s) = E(\frac{e^{iW_t} - e^{-iW_t}}{2i}|W_s).$$

Now for any constant a

$$\begin{aligned} E(e^{aW_t}|W_s) = e^{aW_s}E(e^{a(W_t-W_s)}|W_s) &= e^{aW_s}E(e^{a(W_t-W_s)}) \\ &= e^{aW_s}e^{\frac{a^2(t-s)}{2}}. \end{aligned}$$

Thus

$$E(\sin(W_t)|W_s) = \frac{e^{iW_s}e^{\frac{-(t-s)}{2}} - e^{-iW_s}e^{\frac{-(t-s)}{2}}}{2i} = e^{\frac{-(t-s)}{2}}\sin(W_s).$$

7. Let W_t be a Brownian motion. Compute
   (a) (10 points) $d(W_te^t)$
   Ans: $d(W_te^t) = W_te^tdt + e^tdW_t$.
   (b) (10 points) $d(te^{W_t})$.
   Ans: $d(te^{W_t}) = (e^{W_t} + \frac{1}{2}te^{W_t})dt + te^{W_t}dW_t$.

8. (30 points) Consider a discrete interest rate binomial model as followed:

$$\begin{aligned} r_0 &= 0.02, \\ r_1(u) = 0.03, r_1(d) &= 0.015, \\ r_2(uu) = 0.035, r_2(ud) = 0.02, r_2(dd) &= 0.01. \end{aligned}$$

Suppose the market also consists of a risky asset S where $S_0 = 8$ and at each step the stock price either doubles or halves in value. An interest rate swap is to be entered at time $k = 0$ and expires at time $k = 3$. That is payments made by both parties (to each other) happen at time $k = 1, 2, 3$ where one side pays the fixed rate S and the other side pays the variable rate r_k. Find the swap rate.

Ans: We first compute the risk neutral probability:

$$\begin{aligned} 16 &= \frac{1}{1+0.03}(32q(uu) + 8(1-q(uu))) \\ q(uu) &= 0.3533 \\ q(ud) &= 1-q(uu) = 0.646 \\ 4 &= \frac{1}{1+0.015}(8q(du) + 2(1-q(du))) \\ q(du) &= 0.3433 \\ q(dd) &= 1-q(du) = 0.656 \\ 8 &= \frac{1}{1+0.02}(16q(u) + 4(1-q(u))) \\ q(u) &= 0.3467 \\ q(d) &= 1-q(u) = 0.6533 \end{aligned}$$

The present value (at time $t = 0$) of the fixed leg is:

$$S\sum_{i=1}^{3} B(0,i).$$

The present value (at time $t = 0$) of the variable leg is:

$$1 - B(0,3).$$

The swap rate S satisfies:

$$S\sum_{i=1}^{3} B(0,i) = 1 - B(0,3)$$

Thus

$$S = \frac{1-B(0,3)}{\sum_{i=1}^{3} B(0,i)}.$$

We compute $B(0,3) \approx 0.9446, B(0,2) \approx 0.9610, B(0,1) \approx 0.9803$. Thus

$$S \approx 0.019.$$

9. (30 points) Consider the following model for the short rate in continuous time:

$$dr_t = \mu dt + \sigma \sqrt{r_t} dW_t.$$

This model is not mean-reverting. On the other hand, it is still an affine yield model. That is there exist functions $A(t,T), C(t,T)$ so that the bond price $B(t,T)$ satisfies

$$B(t,T) = e^{-A(t,T)r_t - C(t,T)}.$$

Find the Ordinary differential equations that $A(t,T)$ and $C(t,T)$ satisfy.

Ans:

Let $B(t,T) = u(t,r_t)$. Then $u(t,x)$ satisfies

$$\begin{aligned} -xu + u_t + \mu u_x + \frac{1}{2}\sigma^2 x u_{xx} &= 0 \\ u(T,x) &= 1. \end{aligned}$$

Plugging in

$$u(t,x) = e^{-A(t,T)x - C(t,T)}$$

to the PDE gives

$$-xu - (A_t x + C_t)u - \mu A u + \frac{1}{2}\sigma^2 A^2 x u = 0.$$

That is

$$\begin{aligned} C_t + \mu A &= 0 \\ C(T,T) &= 0. \end{aligned}$$

And

$$\begin{aligned} A_t + 1 - \frac{1}{2}\sigma^2 A^2 &= 0 \\ A(T,T) &= 1. \end{aligned}$$