Math 421
Fall 2016
Midterm exam 2
11/18/16

Name (Print): ____________________

This exam contains 6 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	5	
Total:	105	

1. (20 points) Consider the function $f(x) = x, -\pi < x < \pi$. Find the Fourier series representation of $f(x)$.

   Ans: Here $p = \pi$. Since $f(x)$ is odd, we only need to use sine series representation:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(nx).$$

   Then

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx \\ &= -\frac{2}{n\pi}x\cos(nx)\Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\cos(nx)dx \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

2. (20 points) Consider the function $f(x) = x, 0 < x < \pi$. Find the Fourier *cosine* half-range expansion of $f(x)$.

   Ans:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx).$$

$$\begin{aligned} A_0 &= \frac{2}{\pi}\int_0^{\pi} xdx = \pi \\ A_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx \\ &= -\frac{2}{n\pi}\int_0^{\pi}\sin(nx)dx \\ &= \frac{2}{n^2\pi}(\cos(n\pi) - 1). \end{aligned}$$

   Thus for $n \geq 1$,

$$\begin{aligned} A_n &= \frac{-4}{n^2\pi}, n \text{ odd} \\ &= 0, n \text{ even}. \end{aligned}$$

3. (20 points) Consider the ODE

$$x''(t) - x(t) = f(t), t > 0$$

   where $f(t)$ is $f(t)$ is periodic with period 2π and defined on $[0, 2\pi]$ as followed:

$$\begin{aligned} f(t) &= t, 0 \leq t \leq \pi \\ &= 2\pi - t, \pi \leq t \leq 2\pi. \end{aligned}$$

   Use Fourier series technique to find a particular solution of this ODE.

Ans: Using even-extension (cosine), we have (from the previous problem)

$$f(t) = \pi + \sum_{n=1, n \text{ odd}}^{\infty} \frac{-4}{n^2\pi} \cos(nt).$$

Assuming $x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$ and plugging into the equation we have

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} -a_n(1+n^2)\cos(nt) = \pi + \sum_{n=1, n \text{ odd}}^{\infty} \frac{-4}{n^2\pi} \cos(nt).$$

That is $a_0 = 2\pi$ and $a_n = \frac{4}{n^2\pi(1+n^2)}, n$ odd. $a_n = 0, n$ even.

4. (20 points) Consider the regular Sturm-Liouville problem:

$$\begin{aligned} x'' + \lambda x &= 0, 0 < t < L \\ x'(0) = x'(L) &= 0. \end{aligned}$$

Find all eigenvalues and eigenfunctions of this problem.

Ans: If $\lambda = 0$ then $x(t) = C_1 t + C_2$. The boundary condition implies that $C_1 = 0$ and C_2 can be any value.

If $\lambda = -\alpha^2 < 0$ then $x(t) = C_1 e^{\alpha t} + C_2 e^{-\alpha_t}$. The boundary condition implies that $C_1 = C_2 = 0$.

If $\lambda = \alpha^2 > 0$ then $x(t) = C_1 \cos(\alpha t) + C_2 \sin(\alpha_t)$. The boundary condition implies that $C_2 = 0$. In order for $C_1 \neq 0$ we need $\sin(\alpha L) = 0$ or $\alpha = \frac{n\pi}{L}$.

Thus the eigenvalues are $0, \frac{n\pi}{L}, n \geq 1$. The eigenfunctions are $C, \cos(\frac{n\pi}{L}t)$.

5. (20 points) Consider the heat equation

$$\begin{aligned} u_t &= u_{xx}, 0 < t, 0 < x < L \\ u(0,x) &= x \\ u_x(t,0) = u_x(t,L) &= 0. \end{aligned}$$

(Note the Neumann boundary condition). Find $u(t,x)$.

Ans: Assuming $u(t,x) = T(t)X(x)$. Then $T'(t) = -\lambda T$ and

$$\begin{aligned} X''(x) &= -\lambda X \\ X'(0) &= X'(L) = 0. \end{aligned}$$

From the previous problem

$$\begin{aligned} \lambda^n &= 0, \frac{n\pi}{L}, n \geq 1 \\ X^n(x) &= C, \cos(\frac{n\pi}{L}x). \end{aligned}$$

Also $T^n(t) = e^{-(\frac{n\pi}{L})^2 t}$. Thus

$$u(t,x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 t} \cos(\frac{n\pi}{L}x).$$

The initial condition gives

$$u(0,x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{L}x) = x.$$

We have

$$A_0 = \frac{2}{L}\int_0^L x dx = L.$$

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L x\cos(\frac{n\pi}{L}x) \\ &= -\frac{2}{n\pi}\int_0^L \sin(\frac{n\pi}{L}x) \\ &= \frac{2}{n^2\pi}(\cos(n\pi) - 1). \end{aligned}$$

6. (5 points) Extra credit: Find $\lim_{t\to\infty} u(t,x)$ in the previous heat equation problem.

   Ans: As $n \to \infty, e^{-(\frac{n\pi}{L})^2 t} \to 0$ and thus $u(t,x) \to \frac{L}{2}$.

Scratch (Won't be graded)

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