Math 421 **Name (Print):** ____________________
Fall 2016
Midterm exam 1
10/14/16

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	15	
5	15	
6	20	
7	20	
Total:	100	

1. (10 points) Solve the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 - x_3 &= -1 \\ 3x_1 + x_2 + x_3 &= 5. \end{aligned}$$

Ans: The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & -1 & -1 \\ 3 & 1 & 1 & 5 \end{bmatrix}.$$

The reduced form is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the solution is $x_3 = t, x_2 = 2 - t, x1 = 1$.

2. (10 points) Let

$$
\begin{aligned}
A &= \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\
B &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ b_1 & b_2 & b_3 \\ c_1 - 4a_1 & c_2 - 4a_2 & c_3 - 4a_3 \end{bmatrix}.
\end{aligned}
$$

Given that $det(A) = 1$, find $det(AB)$.

Ans: $det(B) = -1$ since B is obtained from A by multiplying the first row by -1 (which multiplies the determinant by -1) and multiplying row 1 with 4 and add with row 3 (which does not change the determinant). Finally $det(AB) = det(A)det(B) = -1$.

3. (10 points) Let

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

Diagonalize A. That is express A in the form $A = PDP^T$ where D is diagonal and $PP^T = I$.

Ans: $det(A - \lambda I) = (\lambda - 1)(\lambda - 9) - 9 = \lambda^2 - 10\lambda = 0$ if $\lambda = 0, 10$.

If $\lambda = 0$ then $v = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. If $\lambda = 10$ then $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Thus

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$$

4. (15 points) Find

$$\mathcal{L}^{-1}\left(\frac{s^2-1}{(s-1)^3(s+1)^3}\right).$$

Ans:

$$\begin{aligned}
\frac{s^2-1}{(s-1)^3(s+1)^3} &= \frac{(s-1)(s+1)}{(s-1)^3(s+1)^3} = \frac{1}{(s-1)^2(s+1)^2} \\
&= [\frac{1}{2(s-1)} - \frac{1}{2(s+1)}]^2 = \frac{1}{4(s-1)^2} - \frac{1}{2(s-1)(s+1)} + \frac{1}{4(s+1)^2} \\
&= \frac{1}{4(s-1)^2} - \frac{1}{4(s-1)} + \frac{1}{4(s+1)} + \frac{1}{4(s+1)^2}.
\end{aligned}$$

Thus

$$\mathcal{L}^{-1}\left(\frac{s^2-1}{(s-1)^3(s+1)^3}\right) = \frac{1}{4}\Big[te^t - e^t + e^{-t} + te^{-t}\Big].$$

5. (15 points) Solve the initial value problem

$$y' + y = f(t), y(0) = 0$$

where

$$\begin{aligned} f(t) &= 0, 0 \leq t < 1 \\ &= 5, t \geq 1. \end{aligned}$$

Ans: $f(t) = 5\mathcal{U}(t-1)$.
Taking the Laplace transform

$$sY(s) + Y(s) = \frac{5e^{-s}}{s}.$$

Thus

$$Y(s) = \frac{5e^{-s}}{s(s+1)} = 5e^{-s}(\frac{1}{s} - \frac{1}{s+1})$$

Thus

$$y(t) = 5\mathcal{U}(t-1)(1 - e^{-(t-1)}).$$

6. (20 points) Solve the integral equation

$$f(t) = e^{-t} - \int_0^t f(u)e^{t-u}du$$

for $f(t)$.

Ans: Taking the Laplace transform

$$F(s) = \frac{1}{s+1} - \frac{F(s)}{s-1}.$$

That is

$$F(s) = \frac{s-1}{s(s+1)} = \frac{2}{s+1} - \frac{1}{s}.$$

Thus

$$f(t) = 2e^{-t} - 1.$$

7. (20 points) Solve the initial value problem

$$y'' + 9y = \cos(3t), y(0) = 0, y'(0) = 1.$$

for $f(t)$.

Ans: Taking the Laplace transform

$$s^2Y(s) - 1 + 9Y(s) = \frac{s}{s^2+9}.$$

Thus

$$Y(s) = \frac{s}{(s^2+9)^2} + \frac{1}{s^2+9}.$$

That is

$$y(t) = \frac{1}{6}t\sin(3t) + \frac{1}{3}\sin(3t).$$

Scratch (Won't be graded)

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