Math 421
Fall 2016
Final exam
12/20/16

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (two sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	25	
4	20	
5	20	
6	20	
7	25	
8	25	
9	25	
Total:	200	

1. Let

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{bmatrix}.$$

   (a) (10 points) Find the eigenvalues and eigenvectors of A.

   Ans: Since A is diagonal, the eigenvalues are $4, 3, 5$. The corresponding eigenvectors are $[1, 0, 0], [-2, 1, 0], [3, 1, -1]$.

   (b) (10 points) Find the eigenvalues and eigenvectors of A^{-1}. (Remark: It is not necessary to explicitly find A^{-1} for this question).

   The eigenvalues of A^{-1} are $1/4, 1/3, 1/5$. The eigenvectors are the same.

2. (20 points) Let

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{bmatrix}.$$

Find A inverse, if possible. If not, explain why.

$det(A) = -12 + (-6 - 30) - 2(-24) = 0$. Thus A is not invertible.

3. (25 points) Use the Laplace transform to solve the given initial value problem

$$\begin{aligned} y'' + 4y &= e^{-4t}, t > 0 \\ y(0) = 2, y'(0) &= 0. \end{aligned}$$

Ans:

$$s^2Y(s) - 2s + 4Y(s) = \frac{1}{s+4}.$$

Thus

$$Y(s) = \frac{1}{(s+4)(s^2+4)} + \frac{2s}{s^2+4} = \frac{1}{20(s+4)} + \frac{-s+4}{20(s^2+4)} + \frac{2s}{s^2+4}.$$

Thus

$$y(t) = \frac{1}{20}(e^{-4t} - \cos(2t) + 2\sin(2t)) + 2\cos(2t).$$

4. (20 points) Let $f(t)$ be a periodic function with period L, defined for t in $[0, \infty)$ as followed:

$$f(t) = 1, 0 \le t < \frac{L}{2}$$

$$f(t) = 0, \frac{L}{2} \le t < L.$$

Find the Laplace transform of f.

Ans:

$$F(s) = \frac{\int_0^{L/2} e^{-st}dt}{1 - e^{-Ls}} = \frac{1 - e^{-sL/2}}{s(1 - e^{-Ls})}$$

5. (20 points) Let

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

Find the Fourier series representation of $f(x)$.

Ans:

$$f(x) = \frac{A_0}{2} + \sum_{i=1}^{\infty} A_n \cos(nx) + B_n \sin(nx),$$

where

$$A_0 = \frac{1}{\pi}(\int_{-\pi}^{0} 1dx + \int_{0}^{\pi} xdx) = \frac{\pi^2/2 - \pi}{\pi}$$
$$A_n = \frac{1}{\pi}(\int_{-\pi}^{0} \cos(nx)dx + \int_{0}^{\pi} x\cos(nx)dx) = \frac{1-(-1)^n}{\pi n^2}$$
$$B_n = \frac{1}{\pi}(\int_{-\pi}^{0} \sin(nx)dx + \int_{0}^{\pi} x\sin(nx)dx) = \frac{(-1)^n(1-\pi)-1}{\pi n}.$$

6. (20 points) Let

$$f(x) = \cos x, 0 < x < \frac{\pi}{2}.$$

Find the Fourier series cosine half-range expansion of $f(x)$. The following formula maybe helpful:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

Ans:

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} A_n \cos(nx),$$

where

$$A_0 = \frac{4}{\pi}\int_0^{\pi/2} \cos x dx = \frac{4}{\pi}$$
$$\begin{aligned} A_n &= \frac{4}{\pi}\int_0^{\pi/2} \cos x \cos nx dx = \frac{4}{\pi}\int_0^{\pi/2} \cos(n-1)x - \cos(n+1)x dx \\ &= \frac{4}{\pi}\{\sin[(n-1)\pi/2] - \sin[(n+1)\pi/2]\}. \end{aligned}$$

7. (25 points) Consider the heat equation with non-homogeneous Dirichlet boundary conditions

$$\begin{aligned} u_t = u_{xx}, 0 < x < \pi, t > 0 \\ u(0,x) &= 0 \\ u(t,0) = 0, u(t,\pi) &= 1. \end{aligned}$$

Find the explicit solution for $u(t,x)$.

Ans: Let $u(t,x) = v(t,x) + \phi(x)$. We need

$$\begin{aligned} \phi''(x) &= 0, \\ \phi(0) = 0, \phi(\pi) &= 1. \end{aligned}$$

and $v(t,x)$ solves the homogenous heat equation

$$\begin{aligned} v_t &= v_{xx}, 0 < x < \pi, t > 0 \\ v(0,x) &= -\phi(x) \\ v(t,0) = 0, v(t,\pi) &= 1. \end{aligned}$$

Thus

$$\phi(x) = \frac{x}{\pi}.$$

The solution for $v(t,x)$ is

$$v(t,x) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx),$$

where

$$B_n = \frac{2}{\pi^2} \int_0^{\pi} x \sin(nx) dx = \frac{2}{\pi}(-1)^{n+1}.$$

8. (25 points) Consider the non-homogeneous wave equation with Dirichlet boundary conditions

$$\begin{aligned} u_{tt} + g &= u_{xx}, 0 < x < \pi, t > 0 \\ u(0,x) = u_t(0,x) &= 0 \\ u(t,0) = u(t,\pi) &= 0. \end{aligned}$$

(Here g is the gravitation constant). Find the explicit solution for $u(t,x)$.

Ans: Let $u(t,x) = v(t,x) + \phi(x)$. We need

$$\begin{aligned} \phi''(x) &= g, \\ \phi(0) = 0, \phi(\pi) &= 0. \end{aligned}$$

and $v(t,x)$ solves the homogenous heat equation

$$\begin{aligned} v_{tt} &= v_{xx}, 0 < x < \pi, t > 0 \\ v(0,x) &= -\phi(x) \\ v_t(0,x) &= 0 \\ v(t,0) = v(t,\pi) &= 0. \end{aligned}$$

Thus

$$\phi(x) = \frac{g}{2}x^2 - \frac{\pi g}{2}x.$$

The solution for $v(t,x)$ is

$$v(t,x) = \sum_{n=1}^{\infty} A_n \cos(nt)\sin(nx) + B_n \sin(nt)\sin(nx),$$

where

$$\begin{aligned} A_n &= \frac{2}{\pi^2}\int_0^\pi \phi(x)\sin(nx)dx \\ B_n &= 0. \end{aligned}$$

We skip the explicit calculation of A_n. Full credit is given without this calculation.

9. (25 points) Consider the Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0, 0 < x < \pi, 0 < y < \pi \\ u(x,0) = u(0,y) &= 0 \\ u(x,\pi) = u(\pi,y) &= 1. \end{aligned}$$

Find the explicit solution for $u(t,x)$.
Ans: $u(t,x) = u^1(t,x) + u^2(t,x)$ where $u_1(t,x)$ solves

$$\begin{aligned} u^1_{xx} + u^1_{yy} &= 0, 0 < x < \pi, 0 < y < \pi \\ u^1(x,0) = u^1(0,y) &= 0 \\ u^1(x,\pi) = 0, u^1(\pi,y) &= 1 \end{aligned}$$

and $u_2(t,x)$ solves

$$\begin{aligned} u^2_{xx} + u^2_{yy} &= 0, 0 < x < \pi, 0 < y < \pi \\ u^2(x,0) = u^2(0,y) &= 0 \\ u^2(x,\pi) = 1, u^2(\pi,y) &= 0. \end{aligned}$$

We have

$$u^2(x,y) = \sum_{n=1}^{\infty} A_n \sinh(ny)\sin(nx)$$

where

$$A_n = \frac{2}{\sinh(n\pi)}\int_0^\pi \sin(nx)dx = \frac{2(1-(-1)^n)}{\sinh(n\pi)}.$$

By symmetry between x, y we have

$$u^1(x,y) = \sum_{n=1}^{\infty} B_n \sinh(nx)\sin(ny)$$

where

$$B_n = \frac{2}{\sinh(n\pi)}\int_0^\pi \sin(ny)dy = \frac{2(1-(-1)^n)}{\sinh(n\pi)}.$$

Scratch (Won't be graded)

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