Math 485
Fall 2015
Final exam
12/15/2015

Name (Print): ____________________

This exam contains 4 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	40	
2	30	
3	20	
4	20	
5	30	
6	30	
7	30	
Total:	200	

1. (40 points) Let S_t be a stock with the risk neutral dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

S_0 is given. Let $V_T = (S_T - K)^+$ and $V_t = u(t, S_t)$ as the price of the call option on S with expiration T and strike K. Derive the Black-Scholes PDE that u satisfies.

Ans: See class notes.

2. (30 points) Recall the Merton's model for corporate bond: let V_t denote the corporate's total asset with the risk neutral dynamics

$$dV_t = rV_t dt + \sigma V_t dW_t.$$

The bond price $B(t,T)$ for a corportate bond with face value D satisfies $B(T,T) = \min(V_T, D)$. Find the bond's yiled to maturity given that $V_0 = 1$ million dollars, $D = 1.2$ million dollars, $T = 1$ year, $r = 3\%$ per annum, $\sigma = 5\%$. Is it higher than r?

Ans: We have

$$B(0,T) = V_0(1 - N(d+)) + De^{-rT} N(d-),$$

where

$$D(\pm) = \frac{(r \pm \frac{1}{2}\sigma^2)T - \log(\frac{D}{V_0})}{\sigma\sqrt{T}}.$$

Plugging in gives $d+ \approx -3.02$ and $d- \approx -3.06$, which gives $B(0,T) \approx 1$. Thus the yield to maturity is 0.182, which is greater than $r = 0.03$.

3. (20 points) Consider a forward rate agreement for a 6 month period loan with starting date $T_1 = 3$ months from today. Suppose that the zero coupon bond price with face value equal 1 dollar and expiration in 3 and 9 months are 0.98 and 0.96 dollars respectively. What is the forward rate (that is the fixed rate for this loan that would make the FRA free to enter) ?

Ans: $F(0, T_1, T_2) = \frac{1}{3/4 - 1/4}(\frac{0.98}{0.96} - 1) \approx 0.041$

4. (20 points) Consider a 2 period binomial model for the short rate as followed:

$$\begin{aligned} r_0 &= 0.03 \\ r_1(u) = 0.035; r_1(d) &= 0.02 \\ r_2(uu) = 0.04; r_2(ud) = 0.03; r_2(dd) &= 0.01. \end{aligned}$$

Also consider a 2 period binomial stock model with $u = 2$ and $d = 1/2, S_0 = 8$. Find the price for a call option on this stock with expiration $T = 2$ and strike $K = 15$. You can take $\Delta T = 1$(year) in this problem.

Ans: We can compute $q_1(u) = 0.3567, q_1(d) = 0.3467, q_0 = 0.3533$, from which $V_0 = 2.14$.

5. In this problem we will look at the risk neutral pricing concept from a utility function point of view. Consider a game where the player earns u dollars with probability p and d dollars with probability $1-p$. The player is said to be risk neutral if he values this game at $pu + (1-p)d$ dollars. That is he is indifferent between receiving a guaranteed amount that equals to the expectation of the game or receiving the game's random payoff.

   Alternatively, we can model the player's utility function as $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$, for a parameter $\gamma \in \mathbb{R}$. Here u represents the units of utility that the player receives if he has x dollars.

   (a) (5 points) Prove that $u(x)$ is a strictly increasing function in x, irrespective of the sign of γ. This is consistent with the fact that the more money the player has, the happier he is.
   Ans: $u'(x) = e^{-\gamma x} > 0$ thus u is strictly increasing.

   (b) (5 points) Find the inverse function u^{-1} of u.
   Ans: $u^{-1}(x) = \frac{-log(1-\gamma x)}{\gamma}$.

   (c) (5 points) Find $\lim_{\gamma \to 0} u(x)$.
   Ans: Apply L'Hosiptal's rule: $\lim_{\gamma \to 0} u(x) = x$.

   (d) (5 points) The certainty equivalence of a random payoff X for a player with utility function u is defined as $CE = u^{-1}(E(u(X)))$. Note that the certainty equivalence has the unit of money, so it can be used as a candidate for the price of X. Find the certainty equivalence when $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$ and X is a random variable such that $P(X = 1) = 1/2$ and $P(X = -1) = 1/2$.
   Ans: $CE = \frac{-\log(\frac{e^{-\gamma}+e^{\gamma}}{2})}{\gamma}$.

   (e) (10 points) $u(x) = \frac{1-e^{-\gamma x}}{\gamma}$ is referred as the exponential utility. When $\gamma > 0$ we say the player is risk averse, when $\gamma = 0$ we say the player is risk neutral and when $\gamma < 0$ we say the player is risk seeking. Again consider the same pay off random variable X such that $P(X = 1) = 1/2$ and $P(X = -1) = 1/2$. We use the certainty equivalance CE as the price the player is willing to pay for X. Show that for a risk averse player, this price is less than $E(X)$, for a risk neutral player, this price is equal to $E(X)$ and for a risk seeking player this price is greater than $E(X)$. (For convenience you can just consider the case $\gamma = \pm 1, 0$).
   Ans:Plug in $\gamma = \pm 1, 0$ in the answer of the above part we can easily verify the answer since $CE = 0$ when $\gamma = 0$.

6. Consider a market with 2 risky assets S^1, S^2 and the interest rate is 0. We have $S^1_0 = 2, S^2_0 = 3$. There are 3 outcomes for S^1, S^2 at time T, denoted by $\omega_1, \omega_2, \omega_3$. Suppose that $S^1_T(\omega_1) = 3, S^1_T(\omega_2) = 2, S^1_T(\omega_3) = 1$ and $S^2_T(\omega_1) = 4, S^2_T(\omega_2) = S^2_T(\omega_3) = 1$.

(a) (15 points) Is the market abitrage free? If yes, explain why. If no, produce an arbitrage portfolio.

Ans: An arbitrage opportunity would be to long two shares of S^1 and short 2 shares of S^2.

(b) (15 points) Is the market complete? Explain.

The matrix $\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible. Thus the market is complete.

7. let W_t be the standard Brownian motion, $W_0 = 0$. Compute

(a) (10 points) $E(|W_t|)$.

Ans:

$$\begin{aligned} E(|W_t|) &= \frac{2}{\sqrt{2\pi t}} \int_0^\infty x e^{-\frac{x^2}{2}} dx \\ &= \frac{2t}{\sqrt{2\pi t}} \big(-e^{-u}\big|_0^\infty \big) (\text{ by substituting } u = \tfrac{x^2}{2t}) \\ &= \sqrt{\frac{2t}{\pi}}. \end{aligned}$$

(b) (10 points) $E(e^{|W_t|})$.

Ans:

$$\begin{aligned} E(e^{|W_t|}) &= \frac{2}{\sqrt{2\pi t}} \int_0^\infty e^x e^{-\frac{x^2}{2}} dx \\ &= \frac{2e^{\frac{t}{2}}}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{(x-t)^2}{2t}} dx \\ &= 2e^{\frac{t}{2}} P(N(t,t) > 0) \\ &= 2e^{\frac{t}{2}} P(Z > -\frac{t}{\sqrt{t}}) \\ &= 2e^{\frac{t}{2}} N(\frac{t}{\sqrt{t}}). \end{aligned}$$

(c) (10 points) $E((W_t - W_s)(W_t + W_s)|W_r)$ for $r < s < t$.

Ans: Note that $(W_t - W_s)(W_t + W_s) = W_t^2 - W_s^2$. Also $E(W_t^2|W_r) = W_r^2 + t - r$ and $E(W_s^2|W_r) = W_r^2 + s - r$. Thus $E((W_t - W_s)(W_t + W_s)|W_r) = t - s$.