Math 485
Fall 2015
Midterm 2
10/17/2015

Name (Print): ____________________

This exam contains 3 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	40	
2	10	
3	20	
4	20	
Total:	90	

1. (40 points) Let S_t have the following dynamics under a risk neutral measure

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

S_0 is given. Derive the Black-Scholes formula for V_0 where $V_T = (S_T - K)^+$.

Ans: See class notes.

2. (a) (5 points) Let $X_t^1 = \int_0^t e^{\theta s} dWs, \theta$ a constant. Find dX_t^1.
   Ans: $dX_t^1 = e^{\theta t} dW_t$.

   (b) (5 points) Let $X_t^2 = e^{-\theta t}\int_0^t e^{\theta s} dWs$. Find dX_t^2.
   Ans: $dX_t^2 = -\theta X_t^2 + dW_t$.
   Consider the stochastic differential equation (SDE)

$$dS_t = \theta(\mu - S_t)dt + \sigma dW_t,$$

S_0, μ, θ, σ given as constants. Which one of the following processes solves this SDE

$$\begin{aligned} S_t^1 &= S_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + e^{-\theta t}\sigma \int_0^t e^{\theta s} dWs \\ S_t^2 &= S_0 e^{-\theta t} + \sigma \int_0^t e^{\theta s} dWs? \end{aligned}$$

Ans: S_t^1 solves the SDE. We check

$$\begin{aligned} dS_t^1 &= -\theta(S_0 e^{-\theta t} - \mu e^{-\theta t} - e^{-\theta t}\sigma \int_0^t e^{\theta s} dWs)dt + \sigma dW_t \\ &= -\theta(S_t - \mu) + \sigma dW_t. \end{aligned}$$

3. Let W_t be a Brownian motion. For $s < t$ compute

   (a) (5 points) $E(e^{W_t}|W_s)$.
   Ans: Since $W_t|W_s$ has Normal$(W_s, t-s)$ distribution, $E(e^{W_t}|W_s) = e^{W_s + \frac{t-s}{2}}$.

   (b) (5 points) $E\big([\int_0^t W_u dW_u]^2\big)$.
   Ans: $E\big([\int_0^t W_u dW_u]^2\big) = \int_0^t E(W_u^2)du = t^2/2$.

   (c) (5 points) $E\big([\int_0^t u dW_u]^2\big)$.
   Ans: $\frac{t^3}{3}$.

   (d) (5 points) $E(Wt(Ws)^2)$.
   ANs: $E(W_t W_s^2) = E((W_t - W_s)W_s^2) + E(W_s^3) = 0$.

4. (20 points) Consider the partial differential equation

$$\begin{aligned} -u + u_t + xu_x + 2\sigma^2 x^2 u_{xx} &= 0 \\ u(T,x) &= \log(x). \end{aligned}$$

Which one of the following solves the above differential equation? Justify your answer.

$$\begin{aligned} u_1(t,x) &= e^{-(T-t)}\Big(\log(x) + (2 - 1/2\sigma^2)(T-t)\Big) \\ u_2(t,x) &= e^{-(T-t)}\Big(\log(x) + (1 - 2\sigma^2)(T-t)\Big). \end{aligned}$$

Ans: u_2 solves the PDE. We check $u_2(T,x) = \log(x)$ and

$$\begin{aligned} (u_2)_t &= u_2 - e^{-(T-t)}(1-2\sigma^2) \\ (u_2)_x &= e^{-(T-t)}/x \\ (u_2)_{xx} &= -e^{-(T-t)}/x^2. \end{aligned}$$

It's straightforward to plug in and check that u_2 satisfies the PDE.