Math 485
Fall 2015
Midterm 1
10/15/2015

Name (Print): ____________________

This exam contains 4 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. Let X have distribution Uniform[0,Y] distribution, where Y has Exp(1) distribution.
   (a) (10 points) Compute the joint distribution of X, Y and $E(X)$.
   Ans: $f_{XY}(x,y) = f_{X|Y}(x|y)f_Y(y) = e^{-y}, 0 \leq x \leq y, 0 \leq y < \infty$.
   (b) (10 points) Show that

$$E\Big[\big(X - E(X|Y)\big)Y\Big] = E\big(X - E(X|Y)\big)E(Y).$$

   Ans:

$$\begin{aligned} E\Big[\big(X - E(X|Y)\big)Y\Big] &= E\Big(E\Big[\big(X - E(X|Y)\big)Y|Y\Big]\Big) = E\Big(Y\Big[E(X|Y) - E(X|Y)\Big]\Big) = 0 \\ E\big(X - E(X|Y)\big) &= E\Big[E\big(X - E(X|Y)|Y\big)\Big] = E\Big[E(X|Y) - E(X|Y)\Big] = 0. \end{aligned}$$

   Thus they are equal.

2. (a) (10 points) Consider a forward contract based on 2 underlying assets S^1, S^2 with expiration time T and strike price K, that is at time T, the contract holder can purchase 1 share of S^1 and 1 share of S^2 for the price of K dollars. Stock i pay dividend that is continuously reinvested into stock share with dividend rate $q_i, i = 1, 2$. Assume the market is arbitrage free. What is the price of this forward contract at time 0 if the interest rate is r ?
   Ans: Let $x_i, i = 1, 2$ be the number of shares held for stock i. Let y be the cash account. Then we need

$$x_1 e^{q_1 T} S_T^1 + x_2 e^{q_2 T} S_T^2 + y e^{rT} = S_T^1 + S_T^2 - K.$$

   This implies that $x_i = e^{-q_i T}, i = 1, 2$ and $y = -Ke^{-rT}$. Thus

$$V_0 = e^{-q_1 T} S_0^1 + e^{-q_2 T} S_0^2 - Ke^{-rT}.$$

   (b) (10 points) Answer the same question as part (a) but under the assumption that the asset S^1 pays a one time d dollars dividend at time t and asset S^2 pays dividend at rate r_d that is continuously reinvested into S^2.
   Ans: With the same notation as part a:

$$x_1 S_T^1 + x_1 d e^{r(T-t)} + x_2 e^{r_d T} S_T^2 + y e^{rT} = S_T^1 + S_T^2 - K.$$

   Which means

$$(x_1 - 1) S_T^1 + (x_2 e^{r_d T} - 1) S_T^2 + y e^{rT} = -x_1 d e^{r(T-t)} - y e^{rT} - K.$$

   Thus we have $x_1 = 1, x_2 = e^{-r_d T}, y = -(K + de^{r(T-t)})e^{-rT}$ and

$$V_0 = S_0^1 + e^{-r_d T} S_0^2 - (K + de^{r(T-t)})e^{-rT}.$$

3. Consider a market with 2 risky assets S^1, S^2 and the interest rate is 0. We have $S_0^1 = 2, S_0^2 = 3$. There are 3 outcomes for S^1, S^2 at time T, denoted by $\omega_1, \omega_2, \omega_3$. Suppose that $S_T^1(\omega_1) = 3, S_T^1(\omega_2) = 1, S_T^1(\omega_3) = 2$ and $S_T^2(\omega_1) = 4, S_T^2(\omega_2) = 2, S_T^3(\omega_3) = 3$.

   (a) (10 points) Is the market abitrage free? If yes, explain why. If no, produce an arbitrage portfolio.

   Ans: The market is arbitrage free. A risk neutral probability is $q_1 = q_2 = q_3 = 1/3$.

   (b) (10 points) Is the market complete? Explain.

   Ans: The market is not complete because the risk neutral measure is not uniqe. Another possible risk neutral measure is $q_1 = q_2 = 1/4, q_3 = 1/2$.

4. Consider the multiperiod model with $N = 3, S_0 = 8$. From any given period, the stock can either double its price or half its price in the next period (ask for a picture if this description is not clear). Also suppose the interest rate is 0.

   (a) (10 points) Find the price at time period 1, when the stock goes up, that is $V_1(u)$ of a down and out put option with barrier $L = 6$, strike $K = 40$ and expiration $N = 3$. Explicit number is not necessary. You can leave your answer as an expression of summation.

   We have

Path	Pay off
uuu	0
uud	24
udu	24
udd	0
d**	0

   where ** above stands for any combination of u, d. Since $q = 1/3$ we have $V_0 = 48\frac{2}{27}$.

   (b) (10 points) Find the price at time period 0 of the look back option on this stock with expiration $N = 3$. Explicit number is not necessary. You can leave your answer as an expression of summation.

We have

Path	Pay off
uuu	64
uud	32
udu, udd, duu	16
dud, ddu,ddd	8

Thus $V_0 = 64\frac{1}{27} + 32\frac{2}{27} + 16\frac{2+4+2}{27} + 8\frac{4+4+8}{27}$.

5. Consider a game where we toss a fair die. If it's 1 to 5 then we win 1 dollar and have the option to toss the die again. If it's 6 then we lose all the winning we have so far and the game stops (the winning is cumulative, that is if you toss 5 times and the die never shows a 6 then you win 5 dollars).

   (a) (10 points) What is the optimal strategy to play this game?

   Let n be the total winning we currently have. Then our expectation for the next toss given n dollars is $(n+1)\frac{5}{6}$. We will only toss if n is less than or equal to this expectation and stop otherwise. Now

$$n \leq (n+1)\frac{5}{6}$$

if and only if

$$n \leq 5.$$

Thus we still toss if we have 5 dollars or less and stop other wise.

(b) (10 points) What is the fair price to play this game? From the optimal strategy, the maximum winning we can have is 10 dollars, from which we stop. Let V_n be the value of the game corresponding to n dollars we currently have, $n = 0, \cdots, 10$ then we have the following table:

n	V_n
10	10
9	9
8	8
7	7
6	6
5	5

For $n = 0, \cdots, 4$, V_n is calculated by the formula

$$V_n = \frac{5}{6} V_{n+1},$$

because we have $5/6$ chance of gaining 1 dollar and get $n+1$ in our hand and play optimally from stage $n+1$ on and $1/6$ chance of losing everything. Thus we have

$$V_0 = \left(\frac{5}{6}\right)^5 5,$$

which is the fair value of the game.