The Black-Scholes PDE

Math 485

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1 Introduction

Consider the price V_t at time of a European call with strike K and expiration T:

$$V_t = E^Q(e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t).$$

From either the Markov property of S_t or the Indendence Lemma, we see that there is a function $v(t, x)$ (deterministic!) such that for all t,

$$V_t = v(t, S_t).$$

The question is can we derive an equation for $v(t, x)$? The answer is yes, and the equation is a Partial Differential Equation (PDE): an equation connecting the partial derivatives of v in t and x, hence the name.

This equation is of interest because if we can solve it, then to decide V_t we only need to plug in S_t for x. Of course we can decide V_t by taking Expectation via the Independence Lemma, which leads to the Black-Scholes formula. Numerically, this would lead to the pricing by simulation method: we simulate the paths of S_t and summing over the paths as way to approximate the expectation. The pricing of V_t by by figuring out $v(t, x)$ would like to the numerical solution of PDE approach. This provides us with an alternative (and sometimes possibly more powerful) approach to the simulation method described above.

2 Two approaches to derive the Black-Scholes PDE

There are two approaches to derive the Black-Scholes PDE, by constructing either the replicating or the game theory portfolio. We describe both approaches here. The common key to both approaches is the dynamics of a self-financing portfolio: Let π_t be the value of a self financing portfolio consisting of n assets: $S^1, S^2, \cdots, S^n$. (They can be anthing from the underlying assets, the saving account to the financial derivative based on the underlyings). Let $\Delta_t^i, i = 1, \cdots, n$ be the number of shares of S^i in the portfolio at time i. Then

$$d\pi_t = \Delta_t^1 dS_t^1 + \Delta_t^2 dS_t^2 + \cdots + \Delta_t^n dS_t^n.$$

The intuition is since the portfolio is self-financing, the only change in the portfolio value (from one period to another) is from the change in the asset price. You can make this more rigorous by writing from the period $[t_i, t_{i+1}]$:

$$\begin{aligned} \pi_{t_i} &= \Delta_{t_i}^1 S_{t_i}^1 + \Delta_{t_i}^2 dS_{t_i}^2 + \cdots + \Delta_{t_i}^n dS_{t_i}^n \\ \pi_{t_{i+1}} &= \Delta_{t_i}^1 S_{t_{i+1}}^1 + \Delta_{t_i}^2 dS_{t_{i+1}}^2 + \cdots + \Delta_{t_i}^n dS_{t_{i+1}}^n. \end{aligned}$$

Subtracting the two equations, letting $t_i \to t_{i+1}$ we have the dynamics of π_t as described.

3 The game theory approach

3.1 The idea

Recall the game theory portfolio is a portfolio consisting of the underlying S and the derivative V such π_T is a constant in ω. By the no arbitrage condition this forces $e^{rT}\pi_0 = \pi_T$.

This argument can be repeat in a small time interval $[t, t+h]$ to the conclusion that

$$e^{rh}\pi_t = \pi_{t+h}$$

Using Taylor's approximation $e^{rh} \approx 1 + rh$ we have

$$r\pi_t h \approx \pi_{t+h} - \pi_t.$$

Letting h goes to 0, we get

$$d\pi_t = r\pi_t dt.$$

Thus the first key is that **if the portfolio is a game theory portfolio, it has to satisfy**

$$d\pi_t = r\pi_t dt.$$

This is not surprising actually, since it says if a portfolio has deterministic growth, then the growth rate has to be the interest.

The next key is to use the self-financing condition: suppose we hold Δ_t shares of the underlying S and 1 share of V for our game theory portfolio then

$$d\pi_t = \Delta_t dS_t + dV_t.$$

Equating these two equations give

$$\Delta_t dS_t + dV_t = r(\Delta_t S_t + V_t)dt,$$

since $\pi_t = \Delta_t S_t + V_t$. The term dV_t can be expanded by a Taylor expansion to terms involving the partials of the function $v(t, x)$ (recall that $V_t = v(t, S_t)$) with respect to t and x (by applying the Ito's formula). This will lead to a PDE for $v(t, x)$.

3.2 Game theory portfolio in continuous time

The above approach did not address a subtle but crucial point: how can we build a game theory portfolio in continuous time? I.e, how can we choose Δ_t? The answer is not simple. In the 1 period model, we could build a portfolio because by the nature of the model, the stock price only changes at time T and it only has 2 outcomes. In the continuous time, the stock price changes on the interval $[0, T]$ and it takes on a continuum of values. So solving for the number of shares Δ_t of the underlying S_t at every time t cannot be done directly. Instead, we introduce the following idea.

Let S_t be a financial asset in continuous time with the following dynamics:

$$dS_t = \mu_t dt + \sigma_t dW_t.$$

We introduce the following terminology: we call S_t a **risky asset** if $\sigma_t \neq 0$ and we call it a **risk-free** asset if $si_t = 0, \forall t$. That is a risk free asset must have its dynamics as:

$$dS_t = \mu_t dt.$$

Note that a risk-free asset does NOT have to be deterministic. The only requirement is its Brownian motion component is 0. In this way, the bond (or a saving account with variable (in time) and random interest rate) is a risk-free asset.

Back to our game theory portfolio, by applying Ito's formula to find $dV_t = dv(t, S_t)$, we see that $d\pi_t$ consists of a dt and a dW_t term. That is, π_t is an Ito process. For simplicity let us write

$$d\pi_t = \mu_t^\pi dt + \sigma_t^\pi dW_t,$$

where μ_t^π, σ_t^π are some stochastic processes. Observe that by choosing Δ_t carefully, we can control σ_t^π or μ_t^π (for example, possibly making σ_t^π to be 0). If we can do this, then *we can turn π_t into a risk-free asset.*

Why is this important? It is because the ONLY risk-free asset in an arbitrage-free market is the bond, or the saving account. More precisely, we have the following result:

Lemma 3.1. *Let S_t be a risk-free asset. That is suppose*

$$dS_t = \mu_t dt.$$

In addition, suppose μ_t is continuous in t. If the market is arbitrage free, then

$$\mu_t = rS_t, \forall t.$$

Proof. Suppose not, then WLOG we assume $\mu_s > rS_s$ for some s. Since μ_s is continuous, we can find $t > s$ such that $\mu_u > rS_u, u \in [s, t]$. Then

$$S_t = S_s + \int_s^t \mu_u du > S_s + \int_s^t rS_u du.$$

That is

$$S_t > S_s e^{r(t-s)}.$$

Then it is clear that by borrowing money from the bank to invest in S at time s, we will have an arbitrage opportunity. ■

To conclude, *to build a game-theory portfolio in continuous time is equivalent to build a risk-free portfolio.* This will be our approach in deriving the Black-Scholes PDE.

3.3 Derivation of the Black-Scholes PDE

3.3.1 Goal:

To show that under the model

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

the price $v(t, S_t)$ of a European-style derivative that pays $\phi(S_t)$ at time T satisfies

$$\begin{aligned} \frac{\partial}{\partial t}v(t,x) + \frac{\partial}{\partial x}v(t,x)rx + \frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,x)\sigma^2 x^2 - rv &= 0 \\ v(t,x) &= \phi(x). \end{aligned}$$

3.3.2 Ingredients

1. Ito's formula
2. Game-theory portfolio: We hold Δ_t shares of stock at time t and 1 share of V. We choose Δ_t such that the return of the portfolio is "deterministic".
3. No arbitrage principle: If a portfolio π_t satisfies

$$d\pi_t = \mu(t)\pi_t d_t, \tag{1}$$

then we must have $\mu(t) = r$, for all t.

3.3.3 Derivation of Black-Scholes PDE

1. Apply Ito's formula:

$$dV_t = \frac{\partial}{\partial t}v(t, S_t) + \frac{\partial}{\partial x}v(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}v(t, S_t)\sigma^2 S_t^2 dt.$$

Since

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

grouping the d_t and dB_t terms together we have

$$dV_t = \big[\frac{\partial}{\partial t}v(t, S_t) + \frac{\partial}{\partial x}v(t, S_t)rS_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t, S_t)\sigma^2 S_t^2\big]dt + \big[\frac{\partial}{\partial x}v(t, S_t)\sigma S_t\big]dW_t. \tag{2}$$

2.

a. By definition: $\pi_t = \Delta_t S_t + V_t$. By self-financing requirement:

$$\begin{aligned} d\pi_t &= \Delta_t dS_t + dV_t \\ &= \Delta_t \big(rS_t dt + \sigma S_t dW_t \big) + dV_t. \end{aligned}$$

Replace dV_t by (2) and group d_t, dW_t terms again we have

$$\begin{aligned} d\pi_t = \big[\Delta_t r S_t + \frac{\partial}{\partial t} v(t, S_t) + \frac{\partial}{\partial x} v(t, S_t) r S_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 \big] dt \\ + \big[\Delta_t \sigma S_t + \frac{\partial}{\partial x} v(t, S_t) \sigma S_t \big] dW_t. \end{aligned}$$

b. Since π_t is a game theory portfolio, it has the dynamics

$$d\pi_t = r \pi_t dt.$$

Comparing with the above equation, this forces the dW_t term to be 0 or

$$\Delta_t = -\frac{\partial}{\partial x} v(t, S_t).$$

Then

$$\begin{aligned} d\pi_t &= \big[\frac{\partial}{\partial t} v(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, S_t) \sigma^2 S_t^2 \big] dt \\ &= \frac{\big[\frac{\partial}{\partial t} v(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, S_t) \sigma^2 S_t^2 \big]}{\pi_t} \pi_t d_t. \end{aligned}$$

This is in the form of (1) with

$$\mu(t) = \frac{\big[\frac{\partial}{\partial t} v(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, S_t) \sigma^2 S_t^2 \big]}{\pi_t}.$$

Therefore, we conclude that

$$\frac{\big[\frac{\partial}{\partial t} v(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, S_t) \sigma^2 S_t^2 \big]}{\pi_t} = r.$$

But $\pi_t = \Delta_t S_t + v(t, S_t) = -\frac{\partial}{\partial x} v(t, S_t) S_t + v(t, S_t)$. So we have

$$\frac{\partial}{\partial t} v(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, S_t) \sigma^2 S_t^2 = r(-\frac{\partial}{\partial x} v(t, S_t) S_t + v(t, S_t)).$$

In other words

$$\frac{\partial}{\partial t} v(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, S_t) \sigma^2 S_t^2 + r \frac{\partial}{\partial x} v(t, S_t) S_t - r v(t, S_t) = 0.$$

Lastly, since this is true for any value of S_t, replacing S_t by x we have

$$\frac{\partial}{\partial t} v(t, x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} v(t, x) \sigma^2 x^2 + r \frac{\partial}{\partial x} v(t, x) x - r v(t, x) = 0.$$

This is the Black-Scholes PDE.

4 The replicating portfolio approach

4.1 The idea

Recall the replicating portfolio is a portfolio consisting of the underlying S and the saving account y such $\pi_t = V_t$ for any time t. But this forces the dynamics of π_t and V_t to be the same:

$$dV_t = d\pi_t.$$

If we hold Δ_t in S and y_t in the saving account at time t then (recalling that $dy_t = ry_t dt$)

$$dV_t = d\pi_t = \Delta_t dS_t + ry_t dt.$$

The dV_t term can be expanded by Ito's formula as before. The key now is the above equation will become

$$(\text{something1})\ dt + \ (\text{something2})\ dW_t = 0.$$

The second key is we have the **freedom** to choose Δ_t to make the dW_t term to be 0. This forces the dt term to be 0 as well, from which we can derive the PDE. The details are as followed.

4.2 The derivation

1. Apply Ito's formula:

$$\begin{aligned} dV_t &= \frac{\partial}{\partial t}v(t,S_t) + \frac{\partial}{\partial x}v(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,S_t)\sigma^2 S_t^2 dt \\ &= [\frac{\partial}{\partial t}v + rS_t\frac{\partial}{\partial x}v + \frac{1}{2}\frac{\partial^2}{\partial x^2}v\sigma^2 S_t^2]dt + \frac{\partial}{\partial x}v\sigma S_t dW_t. \end{aligned}$$

2. Since

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

we have

$$\begin{aligned} d\pi_t &= \Delta_t(rS_t dt + \sigma S_t dW_t) + ry_t dt \\ &= \Delta_t(rS_t dt + \sigma S_t dW_t) + r(\pi_t - \Delta_t S_t)dt. \end{aligned}$$

Since $V_t = \pi_t$, we conclude $dV_t = dS_t$ and

$$\begin{aligned}
[\frac{\partial}{\partial t}v + rS_t\frac{\partial}{\partial x}v + \frac{1}{2}\frac{\partial^2}{\partial x^2}v\sigma^2 S_t^2]dt + \frac{\partial}{\partial x}v\sigma S_t dW_t &= \Delta_t(rS_t dt + \sigma S_t dW_t) + r(\pi_t - \Delta_t S_t)dt \\
&= \Delta_t(rS_t dt + \sigma S_t dW_t) + r(V_t - \Delta_t S_t)dt \\
&= rvdt + \Delta_t \sigma S_t dW_t.
\end{aligned}$$

The above equation becomes:

$$[\frac{\partial}{\partial t}v + rS_t\frac{\partial}{\partial x}v + \frac{1}{2}\frac{\partial^2}{\partial x^2}v\sigma^2 S_t^2 - rv]dt + (\frac{\partial}{\partial x}v - \Delta_t)\sigma S_t dW_t = 0.$$

Choosing $\Delta_t = \frac{\partial}{\partial x}v(t, S_t)$ the dW_t term is 0. (Note how this term *is exactly the negative of the choice for* Δ_t *in the game theory portfolio.* This is related to the Delta hedging concept). Then the dt term has to be 0 as well, leaving us with

$$\frac{\partial}{\partial t}v + rS_t\frac{\partial}{\partial x}v + \frac{1}{2}\frac{\partial^2}{\partial x^2}v\sigma^2 S_t^2 - rv = 0.$$

This is exactly the same Black-Scholes equation we derived before.

5 Some remarks on Delta hedging

As we see from the derivation, in the game theory portfolio, we have to use $\Delta_t = -\frac{\partial}{\partial x}v(t, S_t)$ and in the replicating portfolio, we have to use $\Delta_t = \frac{\partial}{\partial x}v(t, S_t)$. This is an example of one of the Greeks in Math Finance, which we'll cover later.

The partial derivative of a financial derivative price (or a portfolio price) with respect to the underlying asset price is referred to as the Delta of the financial derivative at time t. It measures the sensitivity of the derivative price with respect to the underlying asset. (Since there are 2 "Deltas" floating around, I will use the Greek symbol Δ_t for the number of assets in the portfolio, and the English word Delta for the concept of partial derivative with respect to the underlying price).

We have seen that it may be desirable for a portfolio to have "stable" return over time (as in the game-theory portfolio). The intuitive idea is to make the Delta of our portfolio to be 0 at all time, so that the portfolio is protected against small change in the underlying asset price in the short run. In particular if we hold Δ_t share of S_t and 1 share of the financial derivative in our portfolio, then

$$\pi_t = \Delta_t S_t + v(t, S_t).$$

Thus (assuming Δ_t is constant)

$$\begin{aligned}\frac{\partial}{\partial S_t}\pi_t &= \Delta_t \frac{\partial}{\partial S_t} S_t + \frac{\partial}{\partial S_t} v(t, S_t) \\ &= \Delta_t + \frac{\partial}{\partial S_t} v(t, S_t).\end{aligned}$$

It follows that the choice $\Delta_t = -\frac{\partial}{\partial x} v(t, S_t)$ will make $\frac{\partial}{\partial S_t}\pi_t = 0$. This choice of Δ_t is referred to as Delta hedging: it makes the Delta of the portfolio value to be 0 in the short run.

Note that the above derivation is NOT rigorous: we assume Δ_t to be constant (or at least independent of S_t) to pass the partial derivative w.r.t S_t through it, only to conclude that it equals $-\frac{\partial}{\partial x} v(t, S_t)$ hence depends on S_t. But the calculation can be heuristically justified as in the short run, Δ_t can be thought of as approximately constant, and all of the above derivation should be looked at only in the approximate sense.

Thus, in practice, at any time moment t one can choose the number of shares Δ_t so that the Delta of the portfolio is approximately 0 for a short time. But then the approximation will no longer be valid after a while (maybe a minute, half an hour etc, say at time $t + \varepsilon$). Then one will need to rebalance the portfolio at that moment to keep the Delta approximately 0 again. One cannot hope to choose Δ for all time t (the buy and hold strategy) while also keep the Delta of the portfolio to be approximately 0 at all time.

6 An example

6.1 Goal:

To show that the price for a cash or nothing derivative: $V_T = \mathbf{1}_{\{S_T \geq K\}}$ at time t, which is

$$\begin{aligned} v(t, S_t) &= e^{-r(T-t)} N(d_2(t, S_t)), \\ d_2(t, S_t) &= \frac{(r - \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}} \end{aligned}$$

satisfies the Black-Scholes PDE:

$$\begin{aligned} \frac{\partial}{\partial t} V + \frac{\partial}{\partial x} V r x + \frac{\partial^2}{\partial x^2} V \sigma^2 x^2 - rV &= 0 \\ v(t, x) = \mathbf{1}_{\{x \geq K\}}. \end{aligned}$$

6.2 Check the terminal condition:

We want to show that

$$\begin{aligned} v(t,x) &= 1 \text{ if } x \geq K \\ &= 0 \text{ if } x < K. \end{aligned}$$

Indeed if $x > K$ then $\frac{K}{x} < 1$ and $\log(\frac{K}{x}) < 0$. Therefore $d_2(T,x) = \infty$ and $N(d_2(T,x)) = 1$.

Similarly $x \leq K$ then $\frac{K}{x} \geq 1$ and $\log(\frac{K}{x}) \geq 0$. Therefore $d_2(T,x) = -\infty$ and $N(d_2(T,x)) = 0$.

6.3 Calculations:

1. Derivatives of $d_2(t,x)$:

$$\begin{aligned} \frac{\partial}{\partial t} d_2(t,x) &= -\frac{r - \frac{1}{2}\sigma^2}{2\sigma\sqrt{T-t}} + \frac{\log(\frac{x}{K})}{2\sigma\sqrt{T-t}^3} \\ &= \frac{1}{2(T-t)}\left[d_2 - \frac{2}{\sigma}(r - \frac{1}{2}\sigma^2)\sqrt{T-t}\right] \\ \frac{\partial}{\partial x} d_2(t,x) &= \frac{1}{x\sigma\sqrt{T-t}} \\ \frac{\partial^2}{\partial x^2} d_2(t,x) &= -\frac{1}{x^2\sigma\sqrt{T-t}}. \end{aligned}$$

2. Derivatives of V:

$$\begin{aligned} \frac{\partial}{\partial t} V &= re^{-r(T-t)}N(d_2(t,x)) + e^{-r(T-t)}\phi_z(d_2(t,x))\frac{\partial}{\partial t} d_2(t,x) \\ &= rV + e^{-r(T-t)}\phi_z(d_2(t,x))\frac{\partial}{\partial t} d_2(t,x) \\ \frac{\partial}{\partial x} V &= e^{-r(T-t)}\phi_z(d_2(t,x))\frac{\partial}{\partial x} d_2(t,x) \\ \frac{\partial^2}{\partial x^2} V &= -e^{-r(T-t)}d_2(t,x)\phi_z(d_2(t,x))(\frac{\partial}{\partial x} d_2(t,x))^2 + e^{-r(T-t)}\phi_z(d_2(t,x))\frac{\partial^2}{\partial x^2} d_2(t,x). \end{aligned}$$

3. Check the cancellations:

a.

$$\begin{aligned} \frac{\partial}{\partial t} V - rV &= e^{-r(T-t)}\phi_z(d_2)\frac{1}{2(T-t)}\left[d_2 - \frac{2}{\sigma}(r - \frac{1}{2}\sigma^2)\sqrt{T-t}\right] \\ &= e^{-r(T-t)}\phi_z(d_2)\frac{1}{2(T-t)}d_2 - e^{-r(T-t)}\phi_z(d_2)\frac{1}{\sigma\sqrt{T-t}}(r - \frac{1}{2}\sigma^2). \end{aligned}$$

b.

$$
\begin{aligned}
\frac{\partial}{\partial x} V r x &= \frac{r e^{-r(T-t)} \phi_z(d_2)}{\sigma \sqrt{T-t}} \\
\frac{1}{2} \frac{\partial^2}{\partial x^2} V \sigma^2 x^2 &= \frac{1}{2} \Big[- \frac{e^{-r(t-t)} \phi_z(d_2) d_2}{T-t} - \frac{e^{-r(T-t)} \phi_z(d_2) \sigma}{\sqrt{T-t}} \Big].
\end{aligned}
$$

It is easy to see that everything cancels out now.