The stock model for continuous time - Pricing Euro style derivative

Math 485

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1 Geometric Brownian motion

1.1 Definition

As motivated in Lecture notes 5a, a natural model for us to use for the underlying process S_t, as the limit of the multiperiod binomial model is

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ, σ are constants, W_t is a BM. We'll use the convention that $\sigma > 0$ (even though S_t would have the same distribution if $\sigma < 0$).

Definition 1.1. *A process S_t following the dynamics*

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

is referred to as a Geometric BM.

The word geometric comes from the intuition that each increment in S_t is a product of S_t with another increment:

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

1.2 Explicit formula

Now we derive an explicit formula for S_t a Geometric BM. Formally dividing both sides of (1) by S_t we have

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

Note that the RHS is free of S_t. Therefore, if we can write the LHS as $df(S_t)$ then we'll be almost done, since then

$$f(S_t) - f(S_0) = \int_0^t df(S_u)du = \int_0^t \mu du + \int_0^t \sigma dW_u,$$

and we can solve for S_t by taking the inverse of f, if possible.

Recalling from classical calculus, the function with a derivative of the form $1/x$ is $\log(x)$. Thus we can try applying Ito's formula to $\log(S_t)$. We have

$$\begin{aligned} d\log(S_t) &= \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt \\ &= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \log(S_t) - \log(S_0) &= \int_0^t (\mu - \frac{1}{2}\sigma^2)du + \int_0^t \sigma dW_u \\ &= (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t. \end{aligned}$$

Therefore,

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

Excerise: Verify that

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

2   Risk neutral measure

2.1   Definition

So far, the dynamics of S_t we gave were under the physical measure P. That is properly we should write

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

under P and

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

under P.

How would we define the risk neutral measure in continuous time? Recalling the intutions that

a. The risk neutral measure must be equivalent to the physical measure P.

b. The risk neutral measure should satisfy

$$V_t = E^Q(e^{-r(T-t)}V_T | \mathcal{F}_t^S),$$

for any financial derivative that pays V_T at time t.

c. The risk neutral price at time t for a financial product that pays S_T at time T (a forward contract on S with zero strike) is S_t.

Thus we see that the proper definition of a risk neutral measure Q is a measure that is equivalent to P such that for any $t \leq T$

$$E^Q(e^{-r(T-t)}S_T | S_t) = S_t.$$

2.2   Conditional expectation for Geometric BM

It is clear that to check whether or not a measure Q is risk neutral, we need to perform a conditional expectation computation of the type

$$E^Q(f(S_T)|S_t),$$

for some function f.

It is reasonable to believe that under the risk neutral measure, S_t is still a geometric Brownian motion, with possibly different μ, σ, W_t. That is it has the following dynamics under Q:

$$dS_t = \mu^Q S_t dt + \sigma^Q S_t dW_t^Q,$$

where μ^Q, σ^Q are constants and W_t^Q a Q-Brownian motion. For our computational purpose, we might as well just perform the expectation under P to have an idea of what we're getting.

Since we're conditioning on S_t, we want to rewrite S_T in terms of S_t, that is

$$E\left\{f(S_T)\Big|S_t\right\} = E\left\{f\left(S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}\right)\Big|S_t\right\}.$$

Note that we have a very familiar situation as what we dealt with in the discrete time: $W_T - W_t$ is independent of S_t. (To see this, note that S_t is a function of W_t by its explicit formula. The independence follows from the independent increment of BM). Therefore, we can apply the Independence Lemma, in the following form:

Theorem 2.1. *Independence Lemma Let X be a RV independent of S_t. Then*

$$E\Big\{ f(X, S_t)|S_t \Big\} = E\Big\{ f(X, x) \Big\}|_{x=S_t}.$$

Thus by the Independence Lemma, we see that

$$E\Big\{ f(S_T)|S_t \Big\} = E\Big\{ f\Big(xe^{(\mu - \frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} \Big) \Big\}|_{x=S_t}.$$

The expectation is taken over the random variable $W_T - W_t$, which has Normal(0,T-t) distribution. In particular, we have

$$E(S_T|S_t) = E\Big\{ xe^{(\mu - \frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} \Big\}|_{x=S_t} = xe^{\mu(T-t)}|_{x=S_t} = S_t e^{\mu(T-t)},$$

where we have used the fact that if X has Normal($0, \sigma^2$) distribution

$$E(e^X) = e^{\frac{1}{2}\sigma^2}.$$

2.3 The dynamics of S_t under the risk neutral measure

From the previous section, we see that if under Q S_t has the following dynamics :

$$dS_t = \mu^Q S_t dt + \sigma^Q S_t dW_t^Q,$$

then

$$E^Q(e^{-r(T-t)}S_T|S_t) = S_t e^{(\mu^Q - r)(T-t)}.$$

Thus if $\mu^Q = r$ under Q then S_t satisfies the risk neutral pricing condition under Q. What about σ^Q ? What can we say about it? It turns out that,σ^Q has to be equal to σ if Q is equivalent to P. The reason will be explained in the Appendix for continuity of exposition.

It turns out that, there is a theorem called the Girsanov theorem, that says there exists such an equivalent measure Q. That is there exists a Q equivalent to P such that under Q

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

where W^Q is a BM under Q. We will take this as given in our further discussion without further dicussing of the proof of Girsanov theorem. Moreover, we will slightly abuse notation and write W_t instead of W_t^Q even when we discuss the distribution of S_t under Q.

3 Pricing of finanical derivative

3.1 The first fundamental theorem of asset pricing in continuous time

Theorem 3.1. *Let S_t be a non-negative continuous stochastic process under P. Then the market consisting of S_t and a saving account is arbitrage free if and only if there exists an equivalent risk neutral measure Q. Moreover, for any financial derivative based on S that makes a payment V_T at time T, the no-arbitrage price of V at time t is*

$$V_t = E^Q(e^{-r(T-t)}V_T|\mathcal{F}_t^S).$$

3.2 Pricing of Euro style derivative

Now suppose V is a Euro style derivative: $V_T = \phi(S_T)$ (that is it is not path-dependent). Then

$$
\begin{aligned}
V_t &= E^Q(e^{-r(T-t)}\phi(S_T)|\mathcal{F}_t^S) \\
&= E^Q\left\{e^{-r(T-t)}\phi\left(S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}\right)\Big|\mathcal{F}_t^S\right\}.
\end{aligned}
$$

Note that S_t is constant given $\mathcal{F}_t^S$. $W_T - W_t$ is independent of $\mathcal{F}_t^S$. That is, we can use the Independence Lemma in the following form:

$$E^Q(f(S_t, W_T - W_t)|\mathcal{F}_t^S) = E^Q(f(x, W_T - W_t))|_{x=S_t}.$$

4 Pricing of the Euro-call option - The Black-Scholes formula

4.1 The Black-Scholes formula

Theorem 4.1. *Suppose that under Q,*

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Then the no-arbitrage price at time 0, V_0 of a European call with strike K and expiration time T satisfies

$$V_0 = S_0 N(d_1) - Ke^{-rT}N(d_2),$$

where $N(d_1) = P(Z \le d_1)$, Z standard Normal and

$$d_1 = \frac{(r + \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}$$

$$d_2 = \frac{(r - \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}$$

The formula for V_0 is referred to as the Black-Scholes formula.

4.2 Outline of the proof of Black-Scholes formula

At the heart of it, the Black-Scholes formula is just computing the expectation of $(X - K)^+$, where X has log normal distribution. However, because of the parameters and the log normal density involved, the computation may seem intimidating for the first time. To help you keep track of what's going on, just keep in mind the big steps that we have to perform in this computation.

1. Computing $E(X - K)^+$, for a continuous RV X with some density $\phi_X(x)$.

The key for this step is to break the function $(x - K)+$ into two parts: $x - K$ for $x > K$ and 0 for $x < K$. That is

$$\int_{-\infty}^{\infty} (x - K)^+ \phi_X(x) = \int_{K}^{\infty} (x - K)\phi_X(x)$$

$$= \int_{K}^{\infty} x\phi_X(x) - \int_{K}^{\infty} \phi_X(x).$$

2. Recognize that $\int_{K}^{\infty} \phi_X(x)$ is just $P(X > K)$.

3. Computing $P(X > K)$ for X having log normal distribution.

The key to this step is just to write X as what it is in distribution: $X = e^Y$, where Y have Normal distribution. Then

$$P(X > K) = P(e^Y > K) = P(Y > \log(K)),$$

and we can use the Z-transform to turn $P(Y > \log(K))$ into a standard Normal calculation.

4. Recognizing that $\int_{K}^{\infty} x\phi_X(x)$ is $E(X\mathbf{1}_{X \ge K})$ where X has log normal distribution (recall the indicator function definition as $\mathbf{1}_E = 1$ if E happens and 0 otherwise).

5. Computing $E(X\mathbf{1}_{X \geq K})$ where X has log normal distribution.

The key to this step is also to write X as what it is in distribution: $X = e^Y$, where Y have Normal distribution. Then

$$E(e^Y \mathbf{1}_{e^Y \geq K}) = E(e^Y \mathbf{1}_{Y \geq \log(K)}) = \int_{\log(K)}^{\infty} e^y \phi_Y(y) dy,$$

where $\phi_Y(y)$ is a Normal density. What we accomplished here is transforming a log-normal expectation computation to a normal expectation computation (because it's complicated to figure out the density of a log-normal distribution).

6. Computing expression of the type $E(e^Y)$, where Y has Normal distribution.

The first key to this step is to recognize that $e^y \phi_Y(y)$, where $\phi_Y(y)$ is a Normal density will be an exponential function, since it will have the form

$$e^y \phi_Y(y) = e^y e^{-\frac{(y-\mu)^2}{2\sigma^2}}.$$

The second key is to **complete the square** for the exponent of the exponential. That is we want to write

$$e^y e^{-\frac{(y-\mu)^2}{2\sigma^2}} = e^{-\frac{(y-\tilde{\mu})^2}{2\tilde{\sigma}^2} + c},$$

for some constants $\tilde{\mu}, \tilde{\sigma}, c$. The point is

$$e^{-\frac{(y-\tilde{\mu})^2}{2\tilde{\sigma}^2}}$$

is (modulo a constant) the density of a Normal$(\tilde{\mu}, \tilde{\sigma})$ so it will integrate to one. The constant e^c will just factor out of the integration.

4.3 Proof of Black-Scholes formula

1. By the risk neutral pricing formula:

$$
\begin{aligned}
V_0 &= E^Q(e^{-rT}V_T) = E^Q(e^{-rT}(S_T - K)^+) = E^Q\left[e^{-rT}\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T} - K\right)^+\right] \\
&= E^Q\left[e^{-rT}\left(S_0 e^X - K\right)^+\right],
\end{aligned}
$$

where X has distribution $N((r-\frac{1}{2}\sigma^2)T, \sigma^2 T)$. Note that $S_0 e^X - K \geq 0$ iff $X \geq \log(\frac{K}{S_0})$. Therefore

$$
\begin{aligned}
V_0 &= \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT}(S_0 e^x - K)\phi_X(x)dx \\
&= \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} S_0 e^x \phi_X(x)dx - \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} K \phi_X(x)dx \\
&:= A - B,
\end{aligned}
$$

where $\phi_X(x) := \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\big(-\frac{(x-\mu)^2}{2\sigma^2 T}\big), \mu := (r - \frac{1}{2}\sigma^2)T$ is the density of X. We will compute A and B separately.

2.

$$
\begin{aligned}
B &= \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} K \phi_X(x)dx = e^{-rT} K P(X > \log(\frac{K}{S_0})) \\
&= e^{-rT} K P\Big(Z > \frac{\log(\frac{K}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma T}\Big) \\
&= e^{-rT} K N(d_2),
\end{aligned}
$$

where

$$
d_2 := \frac{(r - \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}.
$$

3.

$$
\begin{aligned}
A &= \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} S_0 e^x \phi_X(x)dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} S_0 \exp\big(x - \frac{(x-\mu)^2}{2\sigma^2 T}\big)dx.
\end{aligned}
$$

Clearly,

$$
x - \frac{(x-\mu)^2}{2\sigma^2 T} = \frac{2\sigma^2 T x - (x-\mu)^2}{2\sigma^2 T}.
$$

We complete the square in the numerator of the above fraction:

$$
\begin{aligned}
2\sigma^2 T x - (x-\mu)^2 &= -x^2 + 2(\sigma^2 T + \mu)x - \mu^2 = -\big(x - (\sigma^2 T + \mu)\big)^2 + (\sigma^2 T + \mu)^2 - \mu^2 \\
&= -(x - (\sigma^2 T + \mu))^2 + \sigma^4 T^2 + 2\mu\sigma^2 T.
\end{aligned}
$$

Thus

$$x - \frac{(x - \mu)^2}{2\sigma^2 T} = \frac{-(x - (\sigma^2 T + \mu))^2 + \sigma^4 T^2 + 2\mu\sigma^2 T}{2\sigma^2 T}$$

$$= -\frac{(x - (\sigma^2 T + \mu))^2}{2\sigma^2 T} + \frac{1}{2}\sigma^2 T + \mu.$$

Also note that

$$\frac{1}{2}\sigma^2 T + \mu = \frac{1}{2}\sigma^2 T + (r - \frac{1}{2}\sigma^2)T = rT.$$

and

$$\sigma^2 T + \mu = \sigma^2 T + (r - \frac{1}{2}\sigma^2)T = (r + \frac{1}{2}\sigma^2)T.$$

Therefore

$$A = \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} S_0 \exp\left(x - \frac{(x - \mu)^2}{2\sigma^2 T}\right) dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log(\frac{K}{S_0})}^{\infty} S_0 \exp\left(-\frac{(x - (r + \frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}\right) dx$$

$$= S_0 P\left(\tilde{X} \geq \log(\frac{K}{S_0})\right),$$

where $\tilde{X}$ has distribution $N((r + \frac{1}{2}\sigma^2)T, \sigma^2 T)$. Thus

$$A = S_0 P\left(Z \geq \frac{\log(\frac{K}{S_0}) - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$= S_0 N(d_1),$$

where $d_1 = \frac{(r + \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}$.

This finishes the derivation of Black-Scholes formula.

Remark 4.2. *As we mentioned the heart of Black-Scholes formula is just computing the expectation of a log-normal random variable. So if you understand the technique, you can handle much more general Euro style derivative than just the Euro Call option. For example, we can price a Euro style derivative that pays $(S_T^2 - K)^+$ at time T. The details are given at the Appendix.*

4.4 Pricing a Euro call option at time t

Theorem 4.3. *Suppose that under Q,*

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Then the no-arbitrage price at time t, V_t of a European call with strike K and expiration time T satisfies

$$V_t = S_t N(d_1(t)) - Ke^{-r(T-t)} N(d_2(t)),$$

where

$$
\begin{aligned}
d_1(t) &= \frac{(r + \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}} \\
d_2(t) &= \frac{(r - \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}}
\end{aligned}
$$

Proof. By the risk neutral pricing formula and the Independence Lemma:

$$
\begin{aligned}
V_t &= E^Q(e^{-r(T-t)} V_T | S_t) = E^Q(e^{-r(T-t)}(S_T - K)^+ | S_t) \\
&= E^Q\left[e^{-r(T-t)} \left(x e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)} - K \right)^+ \right]\Big|_{x=S_t} \\
&= E^Q\left[e^{-rT} \left(x e^X - K \right)^+ \right]\Big|_{x=S_t},
\end{aligned}
$$

where X has distribution $N((r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$.

Thus we see that we just repeat exactly the same calculation as the original Black-Scholes formula at time $t = 0$, replacing S_0 with S_t and T with $T - t$. The conclusion now follows.

5 Appendix

5.1 $\sigma^Q = \sigma$ between the risk neutral and physical measures

Why does σ^Q have to be equal to σ if Q is equivalent to P? It is because of the following law of iterated logarithm: if W_t is a BM under P then

$$P\left\{ \limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log(t)}} = 1 \right\} = 1.$$

Thus if Q is to be equivalent to P, the Brownian motion W_t cannot be a scaled version of W_t^Q. If $\sigma^Q \neq \sigma$ then essentially we would have W_t^Q as a scaled version of W_t. Then the law of iterated logarithm would require

$$P^Q\left\{ \limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log(t)}} = 1 \right\} = 1,$$

since Q and P are equivalent. But this cannot happen, since we also have

$$P^Q\left\{ \limsup_{t\to\infty} \frac{W_t^Q}{\sqrt{2t\log\log(t)}} = 1 \right\} = P^Q\left\{ \limsup_{t\to\infty} \frac{cW_t}{\sqrt{2t\log\log(t)}} = 1 \right\} = 1,$$

where c is the scaling factor.

Since the two sets

$$\left\{ \limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log(t)}} = 1 \right\}, \left\{ \limsup_{t\to\infty} \frac{W_t^Q}{\sqrt{2t\log\log(t)}} = 1 \right\}$$

are disjoint, it cannot happen that P^Q of these two sets are 1.

5.2 Pricing a Euro style derivative with payment $V_T = (S_T^2 - K)^+$

Lemma 5.1. *Suppose we have a Euro-style derivative that pays $(S_T^2 - K)^+$ at time T where S_t is a geometric BM:*

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Then the no arbitrage at time 0 of this derivative is

$$V_0 = e^{(r+\sigma^2)T}[S_0^2 N(\bar{d}_1) - e^{-rT}\bar{K}N(\bar{d}_2)].$$

where

$$\bar{d}_1 = \frac{(r+2\sigma^2)T - \log(\frac{\bar{K}}{S_0^2})}{2\sigma\sqrt{T}}$$

$$\bar{d}_2 = \frac{(r-2\sigma^2)T - \log(\frac{\bar{K}}{S_0^2})}{2\sigma\sqrt{T}}$$

$$\bar{K} = \frac{K}{e^{(r+\sigma^2)T}}$$

Proof.   From the pricing formula:

$$V_0 = E(e^{-rT}(S_T^2 - K)^+).$$

Note that

$$S_T^2 = S_0^2 \exp((2r - \sigma^2)T + 2\sigma W_T).$$

We need to utilize the Black-Scholes formula, so we want to compare S_t^2 with a process with volatility 2σ. So we consider the process $\bar{S}_t$ where

$$
\begin{aligned}
d\bar{S}_t &= r\bar{S}_t dt + 2\sigma \bar{S}_t dW_t \\
\bar{S}_0 &= S_0^2.
\end{aligned}
$$

That is

$$\bar{S}_t = S_0^2 \exp((r - 2\sigma^2)t + 2\sigma W_t). \tag{1}$$

The Black-Scholes formula gives

$$E(e^{-rT}(\bar{S}_T - K)^+) = \bar{S}_0 N(\bar{d}_1) - Ke^{-rT}N(\bar{d}_2),$$

where

$$\bar{d}_1 = \frac{(r + 2\sigma^2)T - \log(\frac{K}{S_0^2})}{2\sigma\sqrt{T}}$$

$$\bar{d}_2 = \frac{(r - 2\sigma^2)T - \log(\frac{K}{S_0^2})}{2\sigma\sqrt{T}}$$

So in the original computation:

$$
\begin{aligned}
V_0 &= E(e^{-rT}(S_T^2 - K)^+) \\
&= E\left(e^{-rT}(S_0^2 \exp((2r - \sigma^2)T + 2\sigma W_T) - K)^+\right) \\
&= e^{(r+\sigma^2)T} E\left(e^{-rT}(S_0^2 \exp((r - 2\sigma^2)T + 2\sigma W_T) - \bar{K})^+\right) \\
&= e^{(r+\sigma^2)T} E\left(e^{-rT}(\bar{S}_T - \bar{K})^+\right),
\end{aligned}
$$

where $\bar{K} = \frac{K}{e^{(r+\sigma^2)T}}$. Now we have rewritten the formula in the form similar to (1) with $\bar{S}_0 = S_0^2$. Therefore, the conclusion is

$$V_0 = e^{(r+\sigma^2)T}[S_0^2 N(\bar{d}_1) - e^{-rT}\bar{K}N(\bar{d}_2)].$$

■

Proof. Alternative derivation

We can rephrase Black-Scholes formula this way:

$$e^{-rT}E^Q\left((\bar{S}_0 e^{\bar{\mu}T+\bar{\sigma}W_T}-K)^+\right) = e^{-rT}\left(e^{(\bar{\mu}+\frac{1}{2}\bar{\sigma}^2)T}\bar{S}_0 N(d_1)-KN(d_2)\right),$$

where

$$d_1 = \frac{(\bar{\mu}+\bar{\sigma}^2)T-\log\left(\frac{K}{\bar{S}_0}\right)}{\bar{\sigma}\sqrt{T}}$$

$$d_2 = \frac{\bar{\mu}T-\log\left(\frac{K}{\bar{S}_0}\right)}{\bar{\sigma}\sqrt{T}}.$$

Thus in computing

$$V_0 = E(e^{-rT}(S_T^2-K)^+) = E\left(e^{-rT}(S_0^2 e^{(2r-\sigma^2)T+2\sigma W_T})^+\right),$$

we only have to plug in the above with

$$\bar{S}_0 = S_0^2$$
$$\bar{\mu} = 2r-\sigma^2$$
$$\bar{\sigma} = 2\sigma.$$

In this case, we have

$$\begin{aligned}V_0 &= e^{-rT}\left(e^{(2r+2\sigma^2)T}S_0^2 N(d_1)-KN(d_2)\right)\\ &= e^{(r+\sigma^2)T}S_0^2 N(d_1)-Ke^{-rT}N(d_2),\end{aligned}$$

where

$$d_1 = \frac{(2r+3\sigma^2)T-\log\left(\frac{K}{S_0^2}\right)}{2\sigma\sqrt{T}}$$

$$d_2 = \frac{(2r-\sigma^2)T-\log\left(\frac{K}{S_0^2}\right)}{2\sigma\sqrt{T}}.$$

You can check that this is exactly what we got in the first derivation. ∎

5.3 Some further Black-Scholes computation suggestion

You can practice manipulating Black-Scholes formula by considering pricing, that is, find V_t, for the following Euro style derivative, all with expiration T, assuming the Black-Scholes model:

    a. $V_T = \log(S_T)$;

    b. $V_T = S_T^\beta$, β a constant;

    c. $V_T = \mathbf{1}_{S_T \geq K}$, the so-called Binary or cash or nothing option;

    d. $V_T = \mathbf{1}_{K_1 \leq S_T \leq K_2}$, a generalization of the Binary option;

    e. $V_{T_2} = \frac{S_{T_1}}{S_{T_2}}$, $T_1 < T_2$ are two fixed times, the option expires at T_2.

6 Binomial approximation to Black-Scholes price

In this section, we compare the price obtained by Black-Scholes formula with the Binomial tree model, where S_t is a Geometric BM:

$$dS_t \;=\; rS_t dt + \sigma S_t dW_t \tag{2}$$

We will show through an example that the two prices are close to each other. This is not a surprising result, as we showed that the Black-Scholes are indeed obtained from the continuous limit of the Binomial model. You can believe that as the number of periods in the Binomial model grow (so that the length of the period decreases) the price obtained by the Binomial model will converge to the price given by the Black-Scholes model.

6.1 The Black-Scholes price:

For simplicty, we let $r = 0, \sigma = 0.1, T = 1, S_0 = 1000$ and $K = 1000$. Then the Black-Scholes formula for Euro-Call is

$$V_0 = S_0 N(d_1) - K N(d_2)$$

where

$$
\begin{aligned}
d_1 &= \frac{\frac{1}{2}\sigma^2 T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}} \\
&= 0.05
\end{aligned}
$$

Similarly $d_2 = -0.05$. Thus $V_0 = 1000(0.52 - 0.48) = 40$.

6.2 The approximation:

We now divide $[0, 1]$ into $n = 5$ intervals. The discrete approximation to (2) is

$$S_{t_{k+1}} - S_{t_k} = r S_k(t_{k+1} - t_k) + \sigma S_k(W_{t_{k+1}} - W_{t_k})$$

where $k = 0, 1, ..., 5$ and $t_0 = 0, t_1 = 0.2, ..., t_4 = 0.8, t_5 = 1$.

$W_{t_{k+1}} - W_{t_k}$ has distribution $N(0, t_{k+1} - t_k)$. We approximate this by $\sqrt{t_{k+1} - t_k} Y_k$ where

$$
\begin{aligned}
Y_k &= 1 \text{ with probability } \frac{1}{2} \\
&= -1 \text{ with probability } \frac{1}{2}.
\end{aligned}
$$

For short-hand, we will write S_k for S_{t_k}. The evolution equation for S_k becomes

$$S_{k+1} = S_k(1 + \sigma\sqrt{t_{k+1} - t_k} Y_k).$$

Note that this is exactly the Binomial model we have studied before with $X_k = 1 + \sigma\sqrt{t_{k+1} - t_k} Y_k$. Plug in , we have

$$
\begin{aligned}
X_k &= 1.044 \text{ with probability } \frac{1}{2} \\
&= .956 \text{ with probability } \frac{1}{2}.
\end{aligned}
$$

Draw out the binomial tree, we see that the price for Euro Call on S_k with strike 1000 and expiration time $n = 5$ is

$$V_0^b = \left(240 + 5 \times 135 + 10 \times 39.9\right)\frac{1}{2^5} = 41.06$$

This is not a very precise approximation to the Black-Scholes price of course (which gives 40 as in Section 2) but considering we only used 5 steps it is not terrible. The point of this computation is to convince you again that indeed the Geometric Brownian motion can be viewed as the limit of the Binomial tree as the time step gets closer to 0.