The American options

Math 485

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1 Some preliminaries

1.1 Definition

An American call option with strike price K and expiration T on an underlying S gives the holder the right to *choose a time* between 0 and T to *buy* 1 share of S with price K.

An American put option with strike price K and expiration T on an underlying S gives the holder the right to *choose a time* between 0 and T to *sell* 1 share of S with price K.

Remark: The American option allows the possibility of buying and exercising the option *immediately*. Therefore, if we let V_t^A be the price of an American call option with strike K and expiration T, then $V_t^A \geq (S_t - K)^+$ for all t. It is because if $V_t^A < (S_t - K)^+$ then one can buy the option and exercise immediately for a positive profit, which is an arbitrage opportunity. Similarly, the price V_t^A of an American put option also satisfies $V_t^A \geq (K - S_t)^+$ for all t.

1.2 The optimal exercise time

A holder of an American option will judiciously choose a time to exercise the option to maximize his expected profit. Such a time will be referred to as the optimal exercise time. Also note that a prioi, the optimal exercise time is *a random time*. That is for different realizations of the paths of the underlying S, the holder might choose different times to exercise the option.

1.3 Comparison with European option

We have the following easy, but still important observation: Let V_t^A be the (no arbitrage) price of an American call option and V_t^C the (no arbitrage) price of the corresponding European call option purchased at time $t = 0$ with the same strike K and expiry T. That is $V_T^C = (S_T - K)^+$ and the European option holder cannot exercise the option earlier than T. Then $V_t^A \geq V_t^C$. That is the price of an American option is always at least as expensive as its European counterpart. Note that this conclusion is model independent: we do not make any assumption on S_t.

Reason: Suppose $V_t^C > V_t^A$. Then we take a long position (that is we buy 1 share) on the American option and a short position (that is we sell 1 share) on the European option. Then we make a risk free positive profit equals $V_t^C - V_t^A$. At time T, we exercise the American option to close out our short position on the European option. Thus this is an arbitrage opportunity and therefore we must have $V_t^C \leq V_t^A$.

There is another rather surprising result. That is *in the case of a call option, the price of an American option on an asset* **that does not pay dividend** *is equal to the price of a European option for all time*: $V_t^A = V_t^C$, for all t. Thus the optimal exercise time of an American call option will always be the expiration time T. This result also is *model independent.* We present it in the next section.

1.4 American call option

Let $C(t)$ be the price of a European call with strike K expiring at time T. Let $A(t)$ be the price of the corresponding American call. No particular model is put on the price process, except that we assume *no dividends are paid on the asset.* We will use a no-arbitrage argument to show that $C(t) > (S(t) - K)^+$ for all $t < T$, as long as $S(T)$ can fall to either side of K with positive probability. Use this result to show that $A(t) = C(t)$ for all t and that early exercise is never optimal. (It is also helpful to keep this observation in mind: If $A(t) > (S_t - K)^+$ then it is not optimal to exercise the American option at time t since trading the option itself gives higher pay off than exercising the option).

Ans: As long as $P(S(T) > K) > 0$, the price $C(t) > 0$, because the probability of a strictly positive payoff is greater than zero.

If $0 < C(t) \leq (S(t) - K)^+$ at some $t < T$, then $S(t) > K$ and $S(t) \geq C(t) + K$. This would create an arbitrage opportunity. Suppose you short one share of the underlying (that is you borrow $S(t)$ in cash from an agent and pay back one share of S at time T - Another way to think about it is you borrow 1 share of S now and pay it back at time T) and buy the European call for $C(t)$, this leaves you with at least $S(t) - C(t) \geq K$ to invest at the risk-free rate. Since you owe a share of the underlying, your return from this position at T is

$$\begin{aligned}(S(t) - C(t))e^{r(T-t)} + (S(T) - K)^+ - S(T) &= (S(t) - C(t))e^{r(T-t)} - \min\{K, S(T)\} \\ &\geq Ke^{r(T-t)} - \min\{K, S(T)\}\end{aligned}$$

If $r > 0$, this is always positive, and so it yields an arbitrage. If $r = 0$, this payoff is non-negative and strictly positive on the event $S(t) < K$, which we assume happens with positive probability, so again we have an arbitrage. It follows that $C(t) > (S(t) - K)^+$, if $t \leq T$, in order that there is no arbitrage.

The price of the, $A(t)$ of the American call is always greater than or equal to $C(t)$. Thus $A(t) \geq C(t) > (S(t) - K)^+$ when $t < T$. Since the value of the American call is thus always strictly greater than the value of immediate exercise if $t < T$, it is optimal to exercise the American call only at T. It follows then that the American and European call have the same value: $A(t) = C(t)$, for all $t \leq T$.

2 Optimal stopping

Suppose you're the holder of an American option. Your goal is to choose a time to stop (to exercise the option) judiciously to maximize your return. What properties does this time have to satisfy? We have the following observations:

1. The exercise time would be random, rather tha deterministic : it is clear that for different realizations of the asset, you would want to choose different times to exercise the option.

2. Suppose we are currently at time t. The decision, whether or not, to exercise at time t would be based on the past performance of the asset, up to time t. For example, if you hold an American put option, one possible stopping rule is to exercise when the asset price goes beyond a level L, where L is a constant chosen at time 0.

However, the exercise decision cannot be made based on the future information after time t. We say the random time is *a stopping time with respect to the filtration generated by the underlying asset S.* If we denote $\tau(\omega)$ to be the exercise time, then we write $\tau \in \mathcal{F}_t^S$.

3. The best (random) stopping strategy you can make can only be chosen among the stopping time strategies. This is a subtle point to appreciate, as besides stopping

time decision, one can imagine another type of mixed strategy: if we have 2 strategies τ_1, τ_2 then we use τ_1 with probability p and τ_2 with probability $1-p$, and this is done via an independent coin flip. For example, you can do the following: every day you flip a coin and exercise the option the first time the coin flip turns H. This will not be better than making your decision using purely stopping time. The reason is we can show there is an optimal stopping time τ^* that maximizes your return among other stopping times. Thus if you randomized your decision among stopping times, then you miss the "best optimal" stopping time sometimes. More conrete example will be showed below.

4. If we are in a discrete time model, the decision to stop must be made on discrete time points. In other words, the stopping times we referred to above are discrete stopping times. We study some preliminary properties of discrete stopping times in the next section.

3 Discrete stopping time

3.1 Stopping time definition

Definition 3.1. *Let τ be a random variable taking values $\{0, 1, ..., N\}$. We say τ is a stopping time with respect to $\mathcal{F}^S(n)$ if for all $n = 0, 1, ..., N$*

$$\{\tau \leq n\} \quad \in \quad \mathcal{F}^S(n)).$$

Remark 3.2. *Note that the notion of a stopping time is tied to a filtration (similar to the notion of a martingale). It could happen that τ is a stopping time w.r.t a filtration $\mathcal{F}^{S_1}(n)$ but not a stopping time w.r.t another filtration $\mathcal{F}^{S_2}(n)$.*

3.2 Some propreties

1. If τ is a $\mathcal{F}(n)$ stopping time then $\{\tau < n\} = \{\tau \leq n-1\} \in \mathcal{F}(n-1) \subseteq \mathcal{F}(n)$, we have

$$\{\tau \geq n\} \quad = \quad \{\tau < n\}^c \in \mathcal{F}(n)$$

Hence

$$\{\tau = n\} = \{\tau \leq n\} \cap \{\tau \geq n\} \in \mathcal{F}(n).$$

Conversely if $\{\tau = n\} \in \mathcal{F}(n)$ *for all* n then $\{\tau \le n\} = \cup_{i=0}^{n}\{\tau = i\} \in \mathcal{F}(n)$, for all n as well. So we can use either conditions: $\{\tau = n\} \in \mathcal{F}(n)$ or $\{\tau \le n\} \in \mathcal{F}(n)$ as definition for stopping time in discrete time.

2. (*Important*) The event $\{\tau = 0\}$ has probability 0 or 1. The reason is $\{\tau = 0\} \in \mathcal{F}_0^S$, but $\mathcal{F}_0^S$ is the information of the asset up to today, which we assume know. Thus we must know whether or not $\{\tau = 0\}$ with probability 1 (simply put: we must know whether we exercise the option today or not).

3. Let τ_1, τ_2 be stopping times w.r.t. $\mathcal{F}^S(n)$. Then $\min(\tau_1, \tau_2)$ and $\max(\tau_1, \tau_2)$ are stopping times w.r.t $\mathcal{F}^S(n)$.

4 Pricing an American put option in 1 period model

4.1 Mathematical definition

For any time $k, 0 \le k \le N$, let V_k denote the price of an American put option with strike K and expiration N, *provided the option has not been exercised.* This is equivalent to say V_k is the price of an American put option that allows you to choose an exercise time from k to N. From our discussion above, we conclude

$$V_k = \max_{\tau:\ \tau \text{ stopping time taking values from } k \text{ to } N} E^Q\left(e^{-(\tau-k)\Delta T}(K - S_\tau)^+ | \mathcal{F}_k^S\right).$$

This will be **our mathematical definition** of the value of an American option at time k, *if it has not been exercised.*

We call the stopping time that achieves the maximum in the above definition *the optimal stopping time* for V_k, from here on denoted as τ_k^*. We will show that it exists and characterize it in the later section.

4.2 Pricing in 1 period

1. We will now investigate how to solve for $V_k, k = 0, 1, \cdots, N$ in our discrete binomial model. First, suppose that $N = 0$, that is the option exires today, then it is clear that

$$V_0 = (K - S_0)^+.$$

3. Now suppose $N = 1$. Then $V_1 = (K - S_1)^+$. What about V_0? Note that if a stopping time $\tau \in \mathcal{F}_k, k = 0, 1$ then $P(\tau = 0) = 0$ or 1, as well as $P(\tau = 1) = 0$ or 1.

In other words, at time 0 we have 2 choices: exercise the option right away or exercise the option tomorrow. (Any strategy that says with 40% we exercise today and 50% we exercise tomorrow etc. will not be optimal, as we will show).

Let's denote $V_0^1 = E^Q(e^{-r\Delta T}(K - S_1)^+)$ to be the expected return of the option if we go with the strategy exercising tomorrow. Also $V_0^0 = (K - S_0)^+$ is the return we get if we go with the strategy exercising today. Then our decision is clear: exercise today if $V_0^0 \geq V_0^1$ and exercise tomorrow if $V_0^1 > V_0^0$. And the value (the price) of the American option today is

$$V_0 = \max(V_0^1, V_0^0).$$

4. Mathematically we can express the above as followed: define τ^* as

$$\tau^* := \mathbf{1}_{\{V_0^1 > V_0^0\}}.$$

Then (you can check) τ^* is a stopping time (actually it is deterministic) and

$$V_0 = E^Q(e^{-r(\Delta T)\tau^*}(K - S_{\tau^*})^+).$$

5. Why is the mixed strategy (40% today, 60% tomorrow) not optimal? Let τ be a random variable such that $P(\tau = 0) = .4, P(\tau = 1) = .6$ and τ is independent of S. Then the return of this strategy is

$$\begin{aligned} V_0^\tau &= E^Q(e^{-r(\Delta T)\tau}(K - S_\tau)^+) \\ &= E^Q\Big[E^Q(e^{-r(\Delta T)\tau}(K - S_\tau)^+|\tau)\Big] \end{aligned}$$

Since τ is independent of S,

$$\begin{aligned} E^Q(e^{-r(\Delta T)\tau}(K - S_\tau)^+|\tau) &= E^Q(e^{-r(\Delta T)}(K - S_1)^+)\mathbf{1}_{\tau=1} + (K - S_0^+)\mathbf{1}_{\tau=0} \\ &= V_0^1\mathbf{1}_{\tau=1} + V_0^0\mathbf{1}_{\tau=0}. \end{aligned}$$

Thus

$$V_0^\tau = V_0^1 P(\tau = 1) + V_0^0 P(\tau = 0) \leq \max(V_0^1, V_0^0)$$

and hence τ is not an optimal strategy.

5 Pricing an American put option in multi-period model

5.1 An intuitive approach

We want to find a formula for $V_k, 0 \le k \le N$. When $k = N$, the answer is easy: $V_N = (K - S_N)^+$. This is because at the expiration, exercising the option is always no worse than letting it expires worthless.

At $k = N - 1$, the option holder has 2 choices: either exercise immediately, or wait to exercise at time N. Which way is better? If she exercises immediately, she'll get a pay off of $(K - S_{N-1})^+$. If she waits until time N to exercise, the *risk neutral expected payoff* of this choice at time $N - 1$ to her is

$$E^Q(e^{-r\Delta T}(K - S_N)^+|\mathcal{F}^S_{N-1}).$$

She can compare between these two values and decide her strategy depending on which one yields a better payoff. It also follows that at time $N - 1$ the American option is worth

$$V_{N-1} = \max\left\{(K - S_{N-1})^+, E^Q(e^{-r\Delta T}(K - S_N)^+|\mathcal{F}^S_{N-1})\right\}.$$

Generally, at a time k, she has 2 choices: exercise immediately and receive $(K - S_k)^+$, or wait until time $k + 1$ and get the pay off V_{k+1}. How? By following the optimal stopping strategy starting at time $k + 1$ once she is at time $k + 1$. Thus, at time k the American option is worth

$$V_k = \max\left\{(K - S_k)^+, E^Q(e^{-r\Delta T}V_{k+1}|\mathcal{F}^S_k)\right\}.$$

This approach is intuitive, but needs justification on why it is correct. The reason is by definition, V_k is the best risk neutral expected payoff among all stopping time strategies starting at time k. To reach this value, one may imagine the option holder searching among all strategies available to her at time k and choose the optimal one among those. The assertion that

$$V_k = \max\left\{(K - S_k)^+, E^Q(e^{-r\Delta T}V_{k+1}|\mathcal{F}^S_k)\right\}$$

is equivalent to saying **searching optimally from time k to N gives the same payoff as searching optimally from time k to $k+1$, and then search optimally from time $k + 1$ to N.**

By searching optimally from k to $k+1$, and then search optimally from $k+1$ to N we mean choose a strategy that realizes the best payoff comparing if one stops at k or stops at $k+1$, with the payoff at $k+1$ equals to the optimal payoff starting from $k+1$ to N.

But this assertion is not obvious. Stating that it is true amounts to proving *the dynamic programming principle* for V_k.

5.2 The dynamic programming principle

We now develop a formula for V_k. The claim is that

$$V_k = \max\Big\{(K - S_k)^+, E^Q(e^{-r\Delta T} V_{k+1} | \mathcal{F}_k^S)\Big\}. \tag{1}$$

For the binomial model, this translates to

$$V_k(\omega) = \max\Big\{(K - S_k)^+, e^{-rT}[qV_k(\omega u) + (1-q)V_k(\omega d)]\Big\}. \tag{2}$$

Formula (1) **is the basis for the intuitive approach, or the tree pricing method we described above.** Now we prove it.

5.3 Proof of (1)

5.3.1 At $k = N - 1$

Observe that, if τ stopping time taking values from $N-1$ to N, then by a similar argument from the previous section either $P(\tau = N-1|\mathcal{F}_{N-1}^S) = 1$ or $P(\tau = N-1|\mathcal{F}_{N-1}^S) = 1$.

Denote

$$V_{N-1}^N = E^Q(e^{-r\Delta T}(K - S_N)^+ | \mathcal{F}_{N-1}^S).$$

Then by definition, we have

$$V_{N-1} = \max\Big\{(K - S_{N-1})^+, V_{N-1}^N\Big\},$$

which is (1).

Define

$$\tau_{N-1}^* := N\mathbf{1}_{\{V_{N-1}^N > (K - S_{N-1})^+\}} + (N-1)\mathbf{1}_{\{V_{N-1}^N \le (K - S_{N-1})^+\}},$$

then we can check that τ^*_{N-1} is a stopping time and

$$V_{N-1} = E^Q(e^{-r(\Delta T)\tau^*_{N-1}}(K - S_{\tau^*_{N-1}})^+)|\mathcal{F}^S_{N-1}),$$

as in the 1 period model. More importantly,

$$\begin{aligned} V_{\tau^*_{N-1}} &= V_N \mathbf{1}_{\{V^N_{N-1} > (K-S_{N-1})^+\}} + V_{N-1}\mathbf{1}_{\{V^N_{N-1} \leq (K-S_{N-1})^+\}} \\ &= (K - S_N)^+ \mathbf{1}_{\{\tau^*_{N-1}=N\}} + (K - S_{N-1})^+ \mathbf{1}_{\{\tau^*_{N-1}=N-1\}} = (S_{\tau^*_{N-1}} - K)^+. \end{aligned}$$

This is very important: at the optimal exercise time, the value of the option is equal to the excercise value. We will show this is true for general k.

5.3.2 At a general $k \leq N - 1$

We proceed by induction. Suppose (1) is true at step $k + 1$ and the optimal stopping time τ^*_{k+1} exists and the following relation holds:

$$V_{\tau^*_{k+1}} = (S_{\tau^*_{k+1}} - K)^+.$$

Recall that by definition

$$V_k = \max_{\tau:\ \tau \text{ stopping time taking values from } k \text{ to } N} E^Q\Big(e^{-r(\tau-k)\Delta T}(K - S_\tau)^+|\mathcal{F}^S_k\Big).$$

Again observe that, if τ is a stopping time taking values from k to N, then either $P(\tau = k|\mathcal{F}^S_k) = 1$ or $P(\tau > k|\mathcal{F}^S_k) = 1$. The set of these two types of stoping times are mutually exclusive. Therefore, if X is a function of τ

$$\max_{\tau:\ \tau \text{ values from } k \text{ to } N} E(X(\tau)|\mathcal{F}^S_k) = \max\Big\{X_k, \max_{\tau:\ \tau \text{ values from } k+1 \text{ to } N} E(X(\tau)|\mathcal{F}^S_k)\Big\}.$$

Replacing $X(\tau)$ with $e^{-(\tau-k)\Delta T}(K - S_\tau)^+$ we have

$$E(\Big(e^{-(\tau-k)\Delta T}(K - S_\tau)^+|\mathcal{F}^S_k) = E\Big(e^{-r\Delta T}E(\Big(e^{-r(\tau-(k+1))\Delta T}(K - S_\tau)^+|\mathcal{F}^S_k)\Big)$$

and for any τ taking values from $k + 1$ to N

$$\begin{aligned} &E\Big(e^{-r\Delta T}E(\Big(e^{-r(\tau-(k+1))\Delta T}(K - S_\tau)^+|\mathcal{F}^S_k)\Big) \leq \\ &E\Big(e^{-r\Delta T} \max_{\tau:\ \tau \text{ values from } k+1 \text{ to } N} E(\Big(e^{-r(\tau-(k+1))\Delta T}(K - S_\tau)^+|\mathcal{F}^S_k)\Big) = E\Big(e^{-r\Delta T}V_{k+1}\Big). \end{aligned}$$

It follows that

$$V_k \leq \max\Big\{(K - S_k)^+, E^Q(e^{-r\Delta T}V_{k+1}|\mathcal{F}^S_k)\Big\}.$$

On the other hand, it is clear that

$$V_k \geq (K - S_k)^+.$$

Moreover,

$$\begin{aligned} E^Q(e^{-r\Delta T} V_{k+1} | \mathcal{F}_k^S) &= E^Q\Big(e^{-r\Delta T} E^Q[e^{-r\Delta(\tau^*_{k+1}-(k+1))}(K - S_{\tau^*_{k+1}})^+ | \mathcal{F}^S_{k+1}] | \mathcal{F}^S_k \Big) \\ &= E^Q\Big(e^{-r\Delta T(\tau^*_{k+1}-k)}(K - S_{\tau^*_{k+1}})^+ | \mathcal{F}^S_k \Big) \leq V_k. \end{aligned}$$

Thus

$$V_k \geq \max\Big\{ (K - S_k)^+, E^Q(e^{-r\Delta T} V_{k+1} | \mathcal{F}_k^S) \Big\}.$$

Finally denoting $V_k^{k+1} = E^Q(e^{-r\Delta T} V_{k+1} | \mathcal{F}_k^S)$ and define

$$\tau^*_k := \tau^*_{k+1} \mathbf{1}_{\{V_k^{k+1} > (K - S_k)^+\}} + k \mathbf{1}_{\{V_k^{k+1} \leq (K - S_k)^+\}},$$

we see that

$$\begin{aligned} V_{\tau^*_k} &= V_{\tau^* k+1} \mathbf{1}_{\{V_k^{k+1} > (K - S_k)^+\}} + V_k \mathbf{1}_{\{V_k^{k+1} \leq (K - S_k)^+\}} \\ &= (K - S_{\tau^*_{k+1}})^+ \mathbf{1}_{\{\tau^*_k = \tau^*_{k+1}\}} + (K - S_k)^+ \mathbf{1}_{\{\tau^*_k = k\}} = (S_{\tau^*_k} - K)^+, \end{aligned}$$

completing the induction step.