Introduction to basic financial products and pricing (Cont)

Math 485

September 17, 2015

1 Intro to the 1 period model

1.1 Value of a European option

Consider a European call option on an asset S with expiration T and strike price K. From now on, we will *denote the value of a financial product at time t, from the contract holder's point of view,* as V_t. For this Euro call option, we want to figure out what V_T is. Clearly at time T two things can happens: Either $S_T > K$ or $S_T \leq K$. If $S_T > K$, then the holder would exercise the option, to buy 1 share of S at price K. Since the asset is actually worth S_T, he has made a gain of $S_T - K > 0$. If $S_T \leq K$ then he simply does not exercise the option. In that case $V_T = 0$. Thus one can see that

$$V_T = \max(S_T - K, 0).$$

Ntation: For a real number x, we will denote $\max(x, 0)$ as x^+ and $\max(-x, 0)$ as x^-. For example, $5^+ = 5, (-5)^+ = 0, (-5)^- = 5, 5^- = 0$. With this notation, we see that the value of a Euro call option at time T is $V_T^{call} = (S_T - K)^+$ and the value of a Euro put option of exactly the same specification is $V_T^{put} = (S_T - K)^-$.

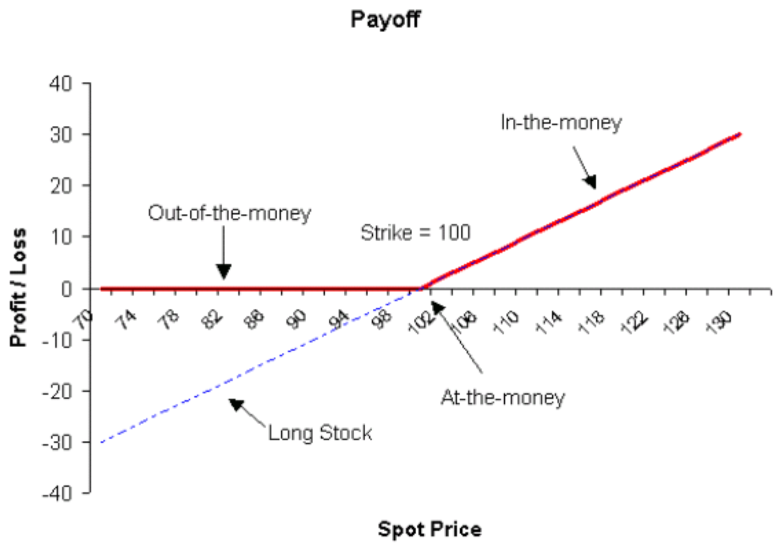

Figure 1: Payoff of a call-option

1.2 An attempt to construct a hedging portfolio for Euro options

Taking the inspiration from our approach of pricing the forward contract, suppose we buy x share of S and hold y dollars in the money market. Can we construct a portfolio that replicates the value of a Euro call option at time T? Clearly the equation that the portfolio value has to satisfy is

$$xS_T + ye^{rT} = (S_T - K)^+,$$

and it has to hold true no matter what S_T is as before. But in this case, this observation doesn't help us to solve for x, y. It is because the function $(S_T - K)^+$ (as a function of S_T) is *not linear in* S_T, so we cannot factor S_T out as before. Thus here we cannot get away with solving for 2 unkowns with 1 equation. We need a second equation. But where should the 2nd equation come from? It is by *positing the specific values* S_T *can take at time* T. Hence the need for a model: the 1 period model.

2 The one period model

2.1 Model specification

The 1 period model is a model about the underlying asset. We posit the followings: there are 2 stages of action, the inital time and the expiration time, where we buy at the initial time and (possibly) exercise the contract at the expiration time. In between we do not do anything (hence the name 1 period). We will denote the initial time as 0 and expiration time as T. The value of the asset at time 0 is S_0, a constant. Its value at time T, S_T is a random variable that can take on 2 values: $S_T = uS_0$ or $S_T = dS_0$ where u and d are positive real numbers. We will also assume the interest rate is $r > 0$. We think of d, u, r as parameters of the model that we can estimate and plug into the model using information about the asset and the interest rate.

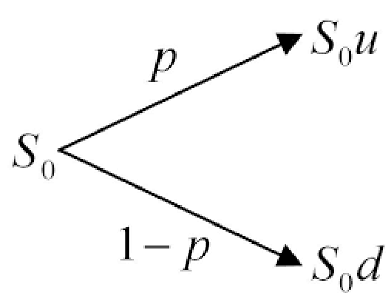

Figure 2: One period binomial model

Note: The textbook uses S_u and S_d instead of uS_0 and dS_0. It is clear that one can go back and forth between these 2 representations. I choose the latter one as it is consistent with the multi-period model. Some of the formulas we'll have below will be slightly different from the book; but if you replace S_u with uS_0 etc. you'll see that they are exactly the same.

2.2 Restriction on d, u, r

When we build a model, we should always do some reality check to see if our model violates any fundamental (or just pure commonsense) principle. In this case, observe that we need to have

$$d < e^{rT} < u,$$

otherwise we'll have an arbitrage opportunity. How to see this? Suppose $d \geq e^{rT}$ then basically the asset is doing better than the money market. One just simply borrow

S_0 from the bank, invest in the stock and will be guaranteed a non-negative profit at time 1. You can argue similarly for $u \leq e^{rT}$.

2.3 Pricing of a financial product using the hedging portfolio

Suppose we have a financial product based on the asset such that, at the expiration time T, if the asset goes up ($S_T = uS_0$) then its value is V_u and if the asset goes down ($S_T = dS_0$) then its value is V_d (V_u and V_d are 2 constants that you can plug in the values for in a specific contract). Note that we do not necessarily require $V_u \geq V_d$. For example if $S_u > K > S_d$ then in a Euro put option $V_u = 0$ and $V_d = K - S_d$. Constructing a replicating portfolio that has x shares of S and y dollars in the money market at time 0, we require:

$$\begin{aligned} xuS_0 + ye^{rT} &= V_u \\ xdS_0 + ye^{rT} &= V_d. \end{aligned}$$

Then easily you see that $x = \frac{V_u - V_d}{S_0(u-d)}$ and $y = \left[V_u - \frac{V_u - V_d}{u-d}u\right]e^{-rT}$. Thus the price for the financial product is

$$V_0 = \frac{V_u - V_d}{u - d} + \left[V_u - \frac{V_u - V_d}{u - d}u\right]e^{-rT}.$$

2.4 Some remarks

1. Note that above we did not just give the price for Euro options, but the price for *ANY* financial product based on S. The only requirement is the correlation: $V_T = V_u$ when $S_T = uS_0$ and $V_T = V_d$ when $S_T = dS_0$. This is very important, without it you can easily see it doesn't work (for example, suppose there are 3 random events $\omega_1, \omega_2, \omega_3$ and $S_T(\omega_1) = S_T(\omega_2) = uS_0, S_T(\omega_3) = dS_0$ but $V_T(\omega_1) = V_u$ and $V_T(\omega_2) = V_T(\omega_3) = V_d$). You should see that there is nothing special about the Euro options (in terms of the math structure). In fact, if we have a method to price Euro options, then that same method can be applied to price any other Euro-style option (that is option that can only be exercised at the terminal time T, which pays a value $f(S_T)$, depending on the outcome of S_T).

2. Our model is very simple (thus not very desirable since it's not close to reality). However, we should be careful about how to make our model more complex. For example if we prescribe that S_1 can take 3 values, then it is not clear whether the above system of equations (which now has 3 equations and 2 unknowns) will have

a solution (it is an overdetermined system). Thus we need to find a way to make our model more realistic, yet still allows for a solution for the pricing. This will be addressed via the multi-period model.

3. It is mentioned that S_T is a random variable, but we **did not give any probability distribution** for it. In fact, it is **not needed at all**, as you have seen. But what if we do have information that allows us to plug in some probability distribution for S_T (say in the case that S is a stock, by observing the performance of the company in the past)? Would this information play some role in determining the price? This is an important, and fundamental question, which we'll address in the next section.

3 Pricing by Expectation

3.1 An example

Consider a company whose stock price is $S_0 = 90$ today. Suppose tomorrow, with probability .9 the stock price is $S_1 = 100$ and with probability .1 the stock price is $S_1 = 80$. Also suppose that the interest rate $r = 0$ and consider a contract that will deliver a share of S to the holder tomorrow, but the payment has to be made today. This is just an example of *a forward contract with strike price* 0. Question: what is the price of such a contract?

Suppose you argue that $E(S_1) = (.9)100 + (.1)80 = 98$, and since there's no interest, the price for the contract is 98 dollars. Is this answer correct?

Before we answer, let's recall that we've already discussed the price of a general forward contract with strike K. The answer is $S_0 - Ke^{-rT}$. Since $K = 0$ the price is $S_0 = 90$, which is *different* from the above expectation. This should give you some suspicion about the above answer.

To explain the discrepancy, also recall the basis for the price $S_0 - Ke^{-rT}$, which is by *constructing a replicating portfolio* and the *no arbitrage principle*. In math finance, whenever you see a proposed price that is different from the no arbitrage price, you can expect that there is an arbitrage opportunity. This is the case here, if the contract is sold for 98. How can we show it?

Actually the work has already been done, since we found the general replicating porfolio for any forward contract before. In this case, if the price of the contract is 98, you would simply sell such a contract , and use the money to buy 1 share of stock at 90. Tomorrow you give the share of stock to the contract holder to close your position, and make an 8 dollar profit *without risk*.

So what is wrong with giving the price by expectation? The short answer is: because this is NOT a game, and the LLN (which is the basis for all pricing by expectation) DOES NOT apply. To see why LLN doesn't apply, note that in this case you only have 1 instance of S_1, which does not fall into the large number of RVs context of the LLN. You may object and ask what if we sell the contracts to many buyers? That would create an instance of a large number of RVs. This is a good point. But notice that even here, the LLN still does not apply, since all of these buyers will face THE SAME S_1 (every one has to see the same stock price tomorrow from the same company). In other words, again there is only 1 RV here, or you can say you have many copies of it, but this does not fulfill the *independent* requirement of the LLN.

Another possible objection: You can say that in this case, the probability that the stock price goes up is very high (there's 90 % chance you make a profit, and only 10 % chance you lose). So in a realistic situation, shouldn't one be very willing to pay a little more for the stock anyway? Maybe you wouldn't pay 98 for the contract, but 90 sounds like a too low value for such a "good" stock. We'll give an answer in the next section.

3.2 A discussion of risk vs expectation

Let's suppose you're offered a game where with 1 % probability you can win 1000 dollars but with 99 % you will lose 9 dollars. You can only play the game ONCE. Would you be willing to play? The expectation of this game is slightly more than 1 dollars. So if you use the expectation as a standard of judgment, then you would definitely play this game. I suspect that in real life, most of us would not play (at least I wouldn't). The reason is most of us are *risk averse*. Since you only play once, there is a very big chance you would lose money. But there're some others who would play the game, since to them paying 9 dollars for a shot of 1000 dollars is worth it (this is what people do with lottery, only that the prize is much bigger and the probability is much smaller). But these are *risk takers*. Again it has nothing to do with expectation. In my opinion, expectation is an irrelevant standard to make the decision in something you only do ONCE. Some may still use it, but then the justification cannot be mathematical, at least not via the LLN.

So to re-emphasize, the price we give for a financial product in this course *will* *always be a no-arbitrage price*, that is it is a price that allows for no arbitrage opportunity based on the product. This, plus the possibility of finding a hedging portfolio, are the only standards we will use to price a product. This is an important fact to keep in mind as we go along.

3.3 The risk neutral distribution

Recall in section (2.3) we found the price of the financial product to be

$$V_0 = \frac{V_u - V_d}{u - d} + \left[V_u - \frac{V_u - V_d}{u - d}u\right]e^{-rT}. \tag{1}$$

Rearranging terms, you'll see that

$$V_0 = e^{-rT}\left[V_u\frac{e^{rT} - d}{u - d} + V_d\frac{u - e^{rT}}{u - d}\right]. \tag{2}$$

Observe that if we denote

$$q = \frac{e^{rT} - d}{u - d}$$

then

$$1 - q = \frac{u - e^{rT}}{u - d}$$

and $0 < q < 1$ (this comes from the specification of the parameters u, d, r in section (2.2)). Thus, q and $1 - q$ is *a probability distribution*. And the above formula can be writen as

$$V_0 = E^Q(e^{-rT}V_T),$$

where E^Q is understood as taking expectation under a distributio that puts weight q on V_u and $1 - q$ on V_d. This particular probability distribution is called *the risk neutral distribution,* or the risk neutral measure (for the 1 period model) (nevermind the word **measure**, you can substitute the word **distribution** wherever you see it in our discussion).

Remark: You may be perplexed, as we just discuss that taking expectation shouldn't be the approach in pricing. Actually there is no contradiction here. What we meant was you shouldn't use expectation as an approach, expecting that the LLN is your justification for pricing. You can view the above formula as a mathematical representation of V_0 (the other representation is formula (1), using the hedging portfolio approach, and does not resemble an expectation at all). In other words, there doesn't have to be a physical interpretation of taking expectation under the risk neutral measure. Indeed the risk neutral measure is one of the more subtle concepts of math finance; and we'll discuss more aspects of it in the next section.

3.4 Remarks about the risk neutral measure

1. In the formula (2), if we use $V_u = uS_0, V_d = dS_0$, then $V_0 = S_0$. This is exactly the same result as we've discussed in section (3.1). In this context, the risk neutral measure has the interpretation that

$$E^Q(e^{-rT}S_T) = S_0. \tag{3}$$

That is, if a person puts the weight of probability on a stock according to the risk neutral measure, then he is indifferent (neutral) on whether he holds a share of stock today or in the future . This is why we call Q the risk neutral measure.

2. We do NOT necessarily live in a risk neutral world. That is in reality, a stock can have a different probability of increasing or decreasing its value from the risk neutral distribution. In fact this is often the case. We call the probability distribution of a stock in the physical world the *objective probability* and denote it by P.

3. Later on, we will use a variation of (3) as a definition of the risk neutral measure. You can see roughly that under the risk neutral measure, the asset has some "nice" behavior (**the expectation of the discounted future value is the present value**). We say a measure is risk neutral if it is *equivalent to the objective probability* and the discounted asset price is a *martingale* under that measure. (We will discuss the notion of martingale in more details later on).

Note: Two probability distributions P and Q are *equivalent* if for any event E, $P(E) = 0$ if and only if $Q(E) = 0$. That is an *improbable* event under one distribution cannot be *probable* under the other distribution. We will explain the reason why we require Q and P to be equivalent below.

4. To repeat, taking expectation under the risk neutral measure should be looked at as a mathematical tool to obtain the *no arbitrage price* for the financial product. The important question is: why does this procedure work? At a preliminary level, you can think of this as giving you a *candidate* for the no arbitrage price (recall the example about the forward contract). Once we have a candidate for a price, we can construct a replicating portfolio and *prove* that it is indeed the right price. But to come up with such a candidate price is not quite simple, especially if the model is complicated.

5. The risk neutral measure is connected with 2 fundamentals theorems of asset pricing. The first says that *there exists a risk neutral measure if and and only if there is no arbitrage opportunity in the market*. The second says that in an arbitrage free market, that is when the risk neutral measure exists, *the risk neutral measure is*

unique if and only if we can build a replicating porfolio for **every** *financial product based on the asset.* Note that the second theorem requires the existence of a risk neutral measure before we can discuss its uniqueness.

Both of these theorems are mathematical results (that is one needs to build a model for the asset and prove these results for that specific model). You can check that in the one-period binomial model we described so far, the risk neutral measure exists, and is unique. (Question: Can you modify the model so that: a. No risk neutral measure exists? b. A risk neutral measure exists, but is not unique ?) The point is that the risk neutral measure is very much connected with no-arbitrage pricing, but to see why it works requires a specific model of the asset and mathematical arguments. Pricing via risk neutral measure is not something one can give a "quick intuition" about why it works.

Remark: Here is why we require the risk neutral measure Q to be equivalent to the objective measure P. The reason is the likelihood of the no arbitrage event only makes sense to be measured under the objective probability P. That is we'd like to make such statements as "the probability that an arbitrage opportunity can happen is 0, in the real world." But to do pricing, we operate under the risk neutral framework. We usually first prove arbitrage cannot happen under the risk neutral measure. But we need to transfer this property back to the real-world in order for our statement to have any significance. The equivalence between P and Q allows us to do so.

6. To re-emphasize, the physical probability is not directly connected to pricing a financial product, except for the requirement that the risk neutral distribution has to be equivalent to it. This is why sometimes we just proceed to compute the risk neutral distribution, without even specifying what the objective probability is.

7. The approach of pricing a financial product from now on will usually take this path: a. compute the risk neutral probability, b. use the risk neutral probability to figure out the risk neutral price (most of the time via a procedure involving taking expectation or conditional expectation under the risk neutral measure), c. verify that we can construct a replicating portfolio using the no-arbitrage price, that is the initial value of the portfolio is exactly the no-aribtrage price.

4 Pricing via the game theory approach

4.1 Preliminary discussion

The last approach of pricing is by the "game theory approach." That is we construct a portfolio consisting of certain shares of the underlying asset and the financial product so that at the expiration time we completely eliminate the risk. (Note: compare this game theory portfolio with the replicating portoflio. A hedging portfolio can only consist of the underlying asset and the money market account, but NOT the financial product. Also note that we can eliminate the risk *exactly because the financial product and the underlying asset is correlated.*) Because we eliminated the risk, the no arbitrage principle says that the value of the portfolio at time T π_T has to be exactly $e^{rT}\pi_0$. This will help us determine what V_0 is. The game theory approach does not have a "realistic" application compared with the replicating portfolio approach. However, it is an interesting idea to apply, which gives us additional understanding about the whole pricing approach of financial products.

4.2 The details

Again consider the 1 period model with a financial product whose termina values can be V_u or V_d. We form a portfolio with x shares of S and y shares of V. The value of the porfolio is as followed:

$$\begin{aligned} \pi_0 &= xS_0 + yV_0 \\ \pi_T &= xS_T + yV_T. \end{aligned}$$

We choose x, y so that

$$xuS_0 + yV_u = xdS_0 + yV_d.$$

That means the value of the portfolio is the same, whether the stock goes up or down. In other words, π_T is a constant. This is what we mean when we say we eliminated the risk (from the portfolio).

There are many ways to choose x, y. One possible solution is

$$y = 1, x = \frac{V_d - V_u}{S_0(u-d)}.$$

Now the key point is that since π_T is a constant, it must follow that $\pi_T = e^{rT}\pi_0$, otherwise there's an arbitrage opportunity (Why?). But then we have

$$\pi_0 = xS_0 + yV_0 = e^{-rT}\pi_T.$$

This means

$$xS_0 + yV_0 = e^{-rT}(xuS_0 + yV_u),$$

Plug in $x = \frac{V_d - V_u}{S_0(u-d)}$ and $y = 1$ gives

$$V_0 = \frac{V_u - V_d}{u-d} + [V_u - \frac{V_u - V_d}{u-d}u]e^{-rT}$$

which is the same as result we obtained before.

5 The fundamental theorems of asset pricing in 1 period model

5.1 Mathematical definition of arbitrage opportunity

Definition 5.1. *An arbitrage opportunity is a* **self-financing** *portfolio* π *such that*

$$\begin{aligned} \pi_0 &= 0, \\ P(\pi_T \geq 0) &= 1 \\ P(\pi_T > 0) &> 0. \end{aligned} \tag{4}$$

Remark:

1. Self-financing means the portfolio's funding is its own money: it cannot have outside source of funding nor can one withdraw (consume) money from the portfolio.

2. Note that the probability used in the definition is the **physical probability**. It makes sense, since intuitively an arbitrage opportunity is a chance to make money without risk. The risk should be measured via the real world probability.

3. Note, however, that since the risk neutral probability P^Q is equivalent to $P, P(\pi_T > 0) = 1$ if and only if $P^Q(\pi_T > 0) = 1$.

4. The portfolio can consist of the underlying asset and the saving account and the financial derivative in consideration.

5. Actually a portfolio does not have to start out at value $\pi_0 = 0$. We can use the following equivalent definition for an arbitrage opportunity:

Definition 5.2. *An arbitrage opportunity is a* **self-financing** *portfolio* π *such that*

$$\begin{aligned} P(e^{-rT}\pi_T \geq \pi_0) &= 1 \\ P(e^{-rT}\pi_T > \pi_0) &> 0. \end{aligned} \tag{5}$$

Exercise: Prove that these definitions are equivalent.

5.2 The fundamental theorems of asset pricing

1. There is no arbitrage opportunity if and only if a risk neutral measure exists.

Let's use the binomial 1 period model. We have seen that this model is always arbitrage free, since we can find the replicating portfolio, *if and only if the condition* $d < e^{rT} < u$ *is satisfied.* We now show this condition is equivalent to the risk neutral measure exists.

Note that

$$E^Q(e^{-rT}S_T) = e^{-rT}S_0(uq + d(1-q)) = S_0$$

and this leads to

$$q = \frac{e^{rT} - d}{u - d}$$

as we mentioned before. Note that $0 < q < 1$ if and only if $d < e^{rT} < u$ in this case. (q cannot be 0 or 1 since in this case, *the risk neutral probability cannot be equivalent to the physical probability*).

2. In an arbitrage free market, the replicating portfolio for **any financial derivative** exists if and only if the risk neutral measure is unique. In this case we say *the market is complete.*

Remark:

a. The key word in the 2nd fundamental theorem of asset pricing is **any derivative**. A consequence of this is if the risk neutral measure is *not* unique, then there must be *some* financial derivative based on the asset that we cannot priced. We give an example where the risk neutral measure is not unique, and the replicating portfolio does not exist: the trinomial model.

b. We need the no arbitrage condition to be satisfied first before we can discuss the completeness of the market. This is mainly used in verifying *the underying asset itself cannot create an arbitrage opportunity,* (as manifested in the condition $d < e^{rT} < u$ in the 1 period binomial model).

5.3 An example in trinomial model

Suppose S_T have three possible outcomes: d, m, u (for down, middle and up). Also for simplicity let $m = e^{rT}$. We will impose $d < e^{rT} < u$ as our usual **necessary condition** for no arbitrage, but *there might be some additional conditions that are required for no arbitrage.* Additional investigation of the model is needed to completely determine what the set of no arbitrage conditions are, in general. Here it turns out that the condition $d < e^{rT} < u$ **is also sufficient**, because of our choice $m = e^{rT}$. For other choices of m, you can see that the no arbitrage conditions need further modification.

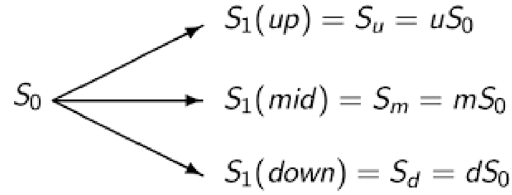

Figure 3: One period trinomial model

And suppose $\min(P(S_T = u), P(S_T = m), P(S_T = d)) > 0$. Then the risk neutral probability Q needs to put positive weight on all these outcomes. So let

$$P^Q(S_T = u) = q_1, P^Q(S_T = e^{rT}) = q_2, P^Q(S_T = d) = 1 - q_1 - q_2.$$

We then require

$$E^Q(e^{-rT} S_T) = e^{-rT} S_0(uq_1 + e^{rT} q_2 + d(1 - q_1 - q_2)) = S_0.$$

That is

$$q_1(u - d) + q_2(e^{rT} - d) = e^{rT} - d.$$

There are several choices for q_1, q_2.

a. If we choose $q_1 = q_2$ then we have

$$q_1 = q_2 = \frac{e^{rT} - d}{u - d + e^{rT} - d}.$$

It follows automatically that $0 < q_1 = q_2 < 1$.

We also need $q_1 + q_2 < 1$ therefore we require

$$\frac{e^{rT} - d}{u - d + e^{rT} - d} < \frac{1}{2}.$$

This is equivalent to $e^{rT} - d < u - d$ or $e^{rT} < u$ as we already have.

b. If we choose $q_2 = 2q_1$ then

$$q_1 = \frac{e^{rT} - d}{u - d + 2(e^{rT} - d)}.$$

This automatically implies that $q_1, q_2 > 0$. We need to guarantee $0 < q_1, q_2, q_3 < 1$. It is enough to require $q_1 + q_2 < 1$ or equivalently $3q_1 < 1$. This is equivalent to

$$\frac{e^{rT} - d}{u - d + 2(e^{rT} - d)} < \frac{1}{3}.$$

But this is equivalent to, again $e^{rT} - d < u - d$ or $e^{rT} < u$.

So you see that **the no arbitrage condition in our 1 period trinomial model is** $d < e^{rT} < u$. However, the risk neutral measure **is not unique** as we have showed there are at least 2 choices for it.

Observe also that **the replicating portfolio does not exist** as we remarked, since the system

$$\begin{aligned} xuS_0 + ye^{rT} &= V_u \\ xe^{rT}S_0 + ye^{rT} &= V_n \\ xdS_0 + ye^{rT} &= V_d \end{aligned}$$

is over-determined. The matrix

$$\begin{bmatrix} uS_0 & e^{rT} & V_u \\ e^{rT}S_0 & e^{rT} & V_n \\ dS_0 & e^{rT} & V_d \end{bmatrix}$$

has REF

$$\begin{bmatrix} uS_0 & e^{rT} & V_u \\ S_0(u - e^{rT}) & 0 & V_u - V_n \\ 0 & 0 & \frac{(V_u - V_n)(u-d)}{u - e^{rT}} - (V_u - Vd) \end{bmatrix},$$

so unless

$$V_u - V_n = (V_u - V_d)\frac{u - e^{rT}}{u - d}, \tag{6}$$

the system cannot have a solution. (Note that for the forward contract $V_T = S_T - K$ we always have a replicating portfolio. It is trivial to check that the forward contract satisfies (6)).

I will leave it for you to check that the trinomial model we give does not allow for arbitrage opportunity.

5.4 No-arbitrage pricing using risk neutral measure

An important consequence of the first fundamental theorem of asset pricing is that if a risk neutral measure Q_1 exists, then **a candidate** for no arbitrage price of a financial derivative that pays V_T at time T is $V_0^1 = E^{Q_1}(e^{-rT}V_T)$. We say a candidate because there may exist another risk neutral measure Q_2 and it is sufficient, for the no-arbitrage condition to hold, to charge $V_0^2 = E^{Q_2}(e^{-rT}V_T)$ for the derivative as well. Thus there is no unique price for the financial product in this case, if our criterion is only the no-arbitrage condition.

To demonstrate, we show that a portfolio consisting of *only the financial derivative* cannot be an arbitrage opportunity. Note that the no arbitrage condition requires that the two following conditions **cannot hold together**

$$\begin{aligned} P(e^{-rT}V_T \geq V_0) &= 1 \\ P(e^{-rT}V_T > V_0) &> 0. \end{aligned}$$

But this is equivalent to requiring that the two following conditions **cannot hold together**, since Q_1 is equivalent to P

$$\begin{aligned} P^{Q_1}(e^{-rT}V_T \geq V_0) &= 1 \\ P^{Q_1}(e^{-rT}V_T > V_0) &> 0. \end{aligned} \tag{7}$$

The conclusion follows because $V_0 = E^{Q_1}(e^{-rT}V_T)$. Indeed if (7) is true, then it must follow that

$$V_0 < E^{Q_1}(e^{-rT}V_T).$$

What about a portfolio consisting of some combination of the financial derivative V, the underlying assset S and the money market y ? As you can see from the above argument, as long as we have the conditions

$$S_0 = E^{Q_1}(e^{-rT}S_T)$$

and

$$y_0 = e^{-rT} y_T$$

satisfied then such portfolio cannot be an arbitratge opportunity. But the first condition is true by the definition of Q_1 being a risk neutral measure and the second is true by the definition of $y_T = e^{rT} y_0$. Thus you see how the existence of an equivalent risk neutral measure implies the no arbitrage condition.

6 Market with more than 1 asset

Observe that in the trinomial model above, the reason why we cannot replicate certain financial product is because we do not *have enough financial assets.* It makes sense that as the financial product "becomes more complex" (in the sense that it has more outcomes), we need more underlying assets to replicate it. In particular, we can imagine a market with 2 underlying assets S^1, S^2. They can go up, stay neutral or go down at time T. Specifically, there are 3 possible outcomes for $S^i_T, i = 1, 2$:

$$\begin{aligned} S^i_T &= u_i S^i_0, \\ S^i_T &= e^{rT} S^i_0, \\ S^i_T &= d_i S^i_0. \end{aligned}$$

For simplicity we can assume that S_1, S_2 move in "synchrony", that is if S^1 goes up, stays neutral or goes down, then S^2 would also go up, stay neutral or go down (This is not so innocent as it seems, *the synchronicity of S^1, S^2 can cause the non existence of the equivalent risk neutral measure*, see examples below).

What are the conditions on u_i, d_i and the synchronicity of S^1, S^2 that would make the model arbitrage free?

We can come up with a financial derivative based on S^1, S^2, for example $V_T = (S^1_T + S^2_T - K)^+$. Can we find a replicating portfolio for **any** V_T (not just this particular example)? The answer is yes, via solving the system

$$\begin{aligned} x_1 u_1 S^1(0) + x_2 u_2 S^2(0) + y e^{rT} &= V_u \\ x_1 e^{rT} S^1(0) + x_2 e^{rT} S^2(0) + y e^{rT} &= V_n \\ x_1 d_1 S^1(0) + x_2 d_2 S^2(0) + y e^{rT} &= V_d \end{aligned}$$

I'll leave you to verify the details. The rule of thumb for asset pricing in discrete model is as followed: suppose a market has n risky assets $S_1, S_2, \cdots, S_n$, and each

$S_k, k = 1, \cdots, n$ has m possible outcomes. Then the market is complete if $n \geq m$. The main idea is if we have more assets than random outcomes then we can replicate any financial derivative.

6.1 Some examples

Suppose $r = 0$. Let $S_0^1 = 200, S_0^2 = 300$ and S_T^1 can take values $400, 200, 100$, S_T^2 can take values $400, 300, 100$. Below we will consider two examples with these set up, just changing the synchronicity of S^1, S^2. You'll see both markets are complete, but one is abitrage free and one is not.

6.1.1 Arbitrage free and complete market

There are 3 possible outcomes $\omega_1, \omega_2, \omega_3$. Then we specify that $S_0^1 = 200$

$$\begin{aligned} S_T^1(\omega_1) &= 400 \\ S_T^1(\omega_2) &= 200 \\ S_T^1(\omega_3) &= 100 \end{aligned}$$

and $S_0^2 = 300$

$$\begin{aligned} S_T^2(\omega_1) &= 300 \\ S_T^2(\omega_2) &= 400 \\ S_T^2(\omega_3) &= 100. \end{aligned}$$

Note that when S^1 goes up, S^2 states neutral and vice versa. We claim that this model is arbitrage free and complete. Indeed you can check that the unique risk neutral probability is

$$P^Q(\omega_1) = \frac{1}{7}; P^Q(\omega_2) = \frac{4}{7}; P^Q(\omega_3) = \frac{2}{7}.$$

Let's apply this to pricing a call option that pays $(S_T^1 + S_T^2 - 400)^+$ at time T. If we hold x_i shares of S^i and y dollars in cash then the matrix system for x_1, x_2, y is

$$\begin{bmatrix} 400 & 300 & 1 & 300 \\ 200 & 400 & 1 & 200 \\ 100 & 100 & 1 & 0 \end{bmatrix},$$

which has REF

$$\begin{bmatrix} 1 & 0 & 0 & 5/7 \\ 0 & 1 & 0 & 3/7 \\ 0 & 0 & 1 & -800/7 \end{bmatrix}.$$

Therefore the price for this option is

$$x_1 S_0^1 + x_2 S_0^2 + y = \frac{5}{7}200 + \frac{3}{7}300 - \frac{800}{7} = \frac{1100}{7},$$

which agrees with the price we get via expectation under risk neutral probability:

$$\frac{1}{7}V_u + \frac{4}{7}V_n + \frac{2}{7}V_d = \frac{1}{7}300 + \frac{4}{7}200 = \frac{1100}{7}.$$

Actually because the LHS of the matrix is always the identity, you can see that any financial product is replicable.

6.1.2 Complete but not arbitrage free market

This time we change the outcomes of S^1, S^2 to $S_0^1 = 200$

$$\begin{aligned} S_T^1(\omega_1) &= 400 \\ S_T^1(\omega_2) &= 200 \\ S_T^1(\omega_3) &= 100 \end{aligned}$$

and $S_0^2 = 300$

$$\begin{aligned} S_T^2(\omega_1) &= 400 \\ S_T^2(\omega_2) &= 300 \\ S_T^2(\omega_3) &= 100. \end{aligned}$$

Note that now S^2 goes up and stays neutral whenever S^1 goes up and stays neutral and vice versa. You can check that the only risk neutral measure we can find is

$$P^Q(\omega_1) = P^Q(\omega_3) = 0; P^Q(\omega_2) = 1.$$

But this is *not equivalent* to the physical measure, assuming that the physical measure puts positive weights on all three outcomes. Thus this means there is arbitrage opportunity for this model. Can you find it?

Surprisingly, this model is still complete. For simplicity again let's consider again the call option that pays $(S_T^1 + S_T^2 - 400)^+$ at time T. If we hold x_i shares of S^i and y dollars in cash then the matrix system for x_1, x_2, y is

$$\begin{bmatrix} 400 & 400 & 1 & 400 \\ 200 & 300 & 1 & 100 \\ 100 & 100 & 1 & 0 \end{bmatrix},$$

which has REF

$$\begin{bmatrix} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -400/3 \end{bmatrix}.$$

The thing to note is the LHS of the matrix again is the identity. Thus any asset is replicable, yet the market is NOT arbitrage free!

7 Asset that pays dividend

In many cases, an asset can pay dividend, either in the form of cash or shares. Then the payment can be made at discrete time points (lump sum payment) or continuously. Eitherway, the stock price will decrease after dividend payment is made. We describe these situations and the effect of dividend payment on a portfolio value.

7.1 Stock dividend

The first form of dividend payment is in a percentage of the stock price, that is *reinvested in to the stock*. This has the overall effect of increasing the number of shares a stock holder have in his portfolio.

Suppose the dividend payment rate is 30 % per annumn. Also suppose that this is a one time payment made at time t. To clarify the situation, we will use t^- to describe the moment right before time t. Then the company makes a one time payment of 30 % S_{t-} in dividend at time t. The stock price S_t becomes $S_t = .7S_{t-}$. If you hold x shares of S then your portfolio value at time t (after the dividend payment) is

$$\pi_t = x(.7S_{t-}) + x(.3S_{t-}) = xS_{t-} = \pi_{t-},$$

so it hasn't changed in value.

Now suppose you use this money to reinvest into the stock, then the number of shares you hold at time t is

$$\tilde{x} = \frac{\pi_t}{S_t} = \frac{xS_{t-}}{0.7S_{t-}} = \frac{x}{.7}.$$

So another way to look at your portfolio value at time t is

$$\pi_t = \frac{x}{0.7}S_t = \frac{1}{1-0.3}xS_t.$$

The above calculation is just an illustration. In the situation that the dividend payment is stock dividend, the reinvesment into the stock *is automatically made.*

Now suppose that the dividend is paid n times over the time interval $[0, T]$, (T is in year), still at the same rate 30% per annum. Then at each time $\frac{0.3T}{n}$ % of the stock price is paid in dividend. You can verify that

$$\pi_T = \left(\frac{1}{1-\frac{0.3T}{n}}\right)^n xS_T.$$

As $n \to \infty$ this approaches

$$\pi_T = e^{0.3T}xS_T.$$

Thus we say if the dividend payment rate is q and the stock is continuously reinvested, then a portfolio consisting of one share of the stock at time 0 is worth $e^{qT}S_T$ at time T. More precisely, 1 share of S has grown to e^{qT} shares of S at time T.

7.2 Cash dividend

The second form of dividend payment is in cash, *that is not automatically reinvested into the stock.* More specifically, suppose a cash amount d is paid out a time t. Then the stock price S_t becomes

$$S_t = S_{t-} - d.$$

A portfolio consisting of x shares of S stays the same in value after the dividend payment:

$$\pi_t = x(S_{t-} - d) + xd = xS_{t-} = \pi_{t-}.$$

Since the cash payment is not automatically reinvested into the stock, the value of this particular portfolio at time T is

$$\pi_T = xS_T + xde^{r(T-t)}.$$

Remark: 1. Technically the cash dividend can also be reinvested into the stock at the discretion of the portfolio holder. But in this case it will come back to the stock dividend case discussed above, just not continuously re-invested. Since we are discussing the one period model, we will assume the portfolio holder does not rebalance their portfolio in between time 0 and time T. The re-balancing situation will be discussed later in the multi-period model.

2. The difference between stock dividend and cash dividend, besides the re-investment issue, is the denomination. Stock dividend is denoted in percentage of stock, and cash is just in dollars.

7.3 Stock price has to decrease after dividend payment

You may wonder if there is a situation where the stock price stays the same after the dividend is paid out (in either form). The answer is no. The reason is there will be arbitrage if this is the case. Suppose a cash dividend d is made at time t. Then if the stock price stays the same, one can simply borrow S_t from the bank to buy 1 share of the asset right before time t, collect the dividend payment d at time t and sell the asset to pay back the loan to the bank. This way one made d dollars in riskless profit. But if the asset price decreases to $S_t - d$ after the dividend payment, then the situation described above won't happen.