# Discrete interest rate model

Math 485

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## 1 Discrete bond model

Consider the binomial multi-period model $0, 1, \cdots, N$. At each time $k$ there are $k+1$ possible outcomes. Consider a bond with maturity $N$ and face value $F$. That is at time $N$ the bond holder will receive $F$ dollars (fixed at time 0). We want to model interest rate $r_k$ and the bond price $B_k$ at time $k$. The length of each period will be denoted as $\Delta T$.

However, what we learned from the previous binomial model cannot be directly used. First we cannot model the bond price following the stock's approach: given $B_0$, define

$$\begin{aligned} B_{k+1} &= uB_k \text{ with probability } p \\ &= dB_k \text{ with probability } 1-p. \end{aligned}$$

The reason is that the bond is financial product that we want to price, so we cannot model it directly. A zero-coupon bond with maturity $N$ is a financial contract that pays the holder 1 dollar at time $N$. Observe that if we follow the stock's binominal modelling approach then there is no way we can get $B_N = 1$. We need to work backward from $B_N = 1$ to figure out $B_k, k = 0, \cdots, n-1$.

The bond is not a financial derivative (it does not derive its value from other underlying). However, the bond price is influenced by interest rate. This suggests that we model the interest rate directly and simply use the pricing formula:

$$B_k = E^Q\left(\frac{1}{1 + r_k \Delta T} B_{k+1} | \mathcal{F}_k\right). \tag{1}$$

We discuss the notation. $\mathcal{F}_k$ represents the state of the world at time $k$, which is known to us at time $k$. We use disrete compounding in this section, hence the discounting factor $\frac{1}{1+r_k\Delta T}$. Interest rate is allowed to be random and varying with time, that is why we need to put $\frac{1}{1+r_k\Delta T}$ inside the expectation.

There is indeed no need for the conditional expectation in the formula (1), as $r_k$ is known at time $k$. However, in general we do need a conditional expectation, as in the following formula

$$B_k = E^Q(\prod_{i=k}^{N} \frac{1}{1+r_i\Delta T} F|\mathcal{F}_k). \tag{2}$$

Here $r_k$ is known at time $k$ but $r_i, k+1 \leq i \leq N$ is not known until time $i$.

$r_k$ is referred to as the short rate. In the context of the discrete model, we understand $r_k$ as the interest rate available for depositing with the money market account (or borrowing) in the period $[k, k+1]$.

If we want to deposit for a longer period (say from time $k$ to $k+2$) there are two options. Either we deposit from time $[k, k+1]$ with interest rate $r_k$ and then roll over to time $[k+1, k+2]$ with interest rate $r_{k+1}$; or we can deposit from time $[k, k+2]$ with the spot rate $R(k, k+2)$. The spot rate $R(k, N)$ is simply the interest that we would earn for depositing during time period $[k, N]$. That is if we deposit 1 dollar at time $k$ and do not withdraw until time $N$ then at time $N$ we would receive $1 + R(k, N)$ dollars. There must be a consistency between these rates; otherwise there would be an arbitrage opportunity. In particular we have

$$B(k, N) = \frac{1}{(1 + R(k, N)\Delta T)^{N-k}} = E^Q(\prod_{i=k}^{N} \frac{1}{1+r_i\Delta T} F|\mathcal{F}_k).$$

The difference is as followed: $R(k, N-k)$ is a random variable that is known at time $k$. $r_k$ is also known at time $k$ but $r_i, k+1 \leq i \leq N$ is not known until time $i$. Observe also that we assume here that $R(k, N)$ is a compound interest rate that is compounded at the end of each period. If $R(k, N)$ is a simple interest rate the above equation would read

$$B(k, N) = \frac{1}{(1 + R(k, N)(N-k)\Delta T)} = E^Q(\prod_{i=k}^{N} \frac{1}{1+r_i\Delta T} F|\mathcal{F}_k).$$

That is, the interest is only compounded once at the maturity time $N$. We need to rely on the context to distinguish between simple and compound interest rate.

## 2 Stochastic short rate model

Now we turn to modelling $r_k, k = 0, \cdots, N$. One important observation here is that we need to model $r_k$ directly under the risk neutral measure $Q$. That is we need to take the risk neutral measure $Q$ as given exogenously (maybe from working with other financial dirvatives) and model $r_k$ under this measure. The reason is if our market only consists of bond then there is no reference underlying, such as a stock, for us to define $Q$ upon. In the lack of such reference product, we take our default risk neutral measure as $q = 1 - q = 1/2$ for example.

We need to be careful on how to model $r_k$. We can use the binomial approach

$$\begin{aligned} r_{k+1} &= ur_k \text{ with probability } q \\ &= dr_k \text{ with probability } 1-q; \end{aligned}$$

but it make not be realistic as this allows for the interest rate to grow quite large in some particular event (in contrast to the real world observation that interest rate tends to stay around some fixed level, say right now close to 0%, for a fixed period of time). If we want to avoid specifying a formula, we can just simply 'fill in the nodes of the binomial tree for the intrest rate value at that node. For example if $N = 3$ we can specify

$$r_3(uuu) = 0.03, r_3(udu) = 0.025, r_d(ddu) = 0.0125, r_d(ddd) = 0.025,$$

and so on for $r_2, r_1, r_0$. Once we have the interest rate tree, we can use formula (1) to figure out the bond price with all maturities up to $N$.

## 3 The money market account

If we deposit 1 dollar at time $k$, we will earn $1 + r_k\Delta T$ dollars at time $k + 1$. The money market account $M_k$ is the value at time $k$ of the account that deposits 1 dollar at time 0. That is

$$M_k = \prod_{i=0}^{k-1}(1 + r_i\Delta T).$$

Observe that even though $M_k$ is random, $M_{k+1}$ is known at time $k$ (since $r_k$ is determined at time $k$). In this way we say the money market account processs $M_k$ is predictable.

The money market account can be used in the pricing formula as followed: for $0 \leq i < j \leq N$

$$B_i = E^Q(\prod_{k=i}^{j-1} \frac{1}{1 + r_k \Delta T} B_j | \mathcal{F}_i).$$

Observe that $\prod_{k=i}^{j-1} \frac{1}{1+r_k \Delta T} = \frac{M_j}{M_i}$ and $M_i$ is known at time $i$ we can write the pricing formula succintly as

$$M_i B_i = E^Q(M_j B_j | \mathcal{F}_i).$$

# 4 Option pricing in stochastic interest rate model

Consider a binomial tree model where we have the underlying $S$ that follows the model

$$\begin{aligned} S_{k+1} &= uS_k \text{ with probability } p \\ &= dS_k \text{ with probability } 1 - p. \end{aligned}$$

Suppose we have a call option on $S$ with strike $K$ and expiration $N$. That is $V_N = (S_N - K)^+$. We want to find $V_0$, given a stochastic interest rate model. To this end, suppose that a stochastic interest tree has already be given as in section (2). Note though that we cannot specify the risk neutral measure $q = 1/2$ by default as in that section. The reason is because we have a reference underlying here. So we need to use the definition:

$$S_k = E^Q(\frac{1}{1 + r_k \Delta T} S_{k+1} | S_k).$$

This implies

$$(1 + r_k \Delta T) S_k = q_k u S_k + (1 - q_k) d S_k,$$

or

$$(1 + r_k \Delta T) = q_k u + (1 - q_k) d.$$

One can solve for $q_k$ from there. Note something subtle here. Not only that $q$ depends on $k$ as $r$ depends on $k$ but it also depends on the random event that happens at time $k$. That is the $q_k$ in the above equation is a conditional probability. It is the risk neutral probability that given an even $\omega_k$ at time $k$, the next even twill be $\omega_k u$:

$$P^Q(\omega_k u|\omega_k) = q_k(\omega_k).$$

Once we can fill out all the risk neutral probabilities this way we can proceed to compute the option price via the backward pricing method as we discussed before.

# 5 Coupon paying bond and future cash flow

So far we have discussed zero-coupon bond. That is bond with only a face value, here denominated as 1 dollar, which is paid at the maturity time $N$. More generally, one can consider a coupon paying bond with coupon rate $C$ and face value $F$. That is at each period $k = 0, 1, \cdots, N$ (determined by the bond, usually annually or semi-annually) the bond holder receives $C\%$ of the face value and receive the face value $F$ dollars at the maturity time $N$. The pricing of such a coupon paying bond, given an interest rate tree, is similar to the method discuss above. Indeed, we can look at the zero-coupon bond as a future income stream that pays us $C_k = \frac{C}{100}F$ at time $k$. We can compute the present-value $C_k^0$ of a future payment $C_k$ at time $k$ by the risk neutral pricing formula

$$C_k^0 = E^Q(\frac{1}{M_k}C_k).$$

Then the present value of the zero-coupon bond is the sum of the present values in the future income stream:

$$C_0 = \sum_{k=0}^{N} C_0^k.$$

In the case of a coupon bond with a fixed coupon rate (say 5%) there is even a more straightforward method to compute its present value without the need for modeling the interest rate. Indeed, the present value of 1 dollar paid at time $k$ is just $B(0, k)$, the price of a zero coupon bond that paids 1 dollar at time $k$. Thus in the above formula, $C_k^0 = B(0, k)C_k = B(0, k)\frac{C}{100}F$.

# 6 Yield to maturity

Precisely speaking, bond price is a function of two time variables: the current time $k$ and the maturity time $N$. Thus we write $B(k, N)$ for the price at time $k$ of a bond with maturity $N$. For a present value of time, say $k = 0$, we can discuss the price of the bond that matures at time $N = 1, 2, \cdots$. Term structure refers to the concept of fixing a present time and varies the maturity of a fixed income security, typically a bond. The term structure reflects the public opinion on how the interest rate will behave in the future.

A particular quantity of interest is the bond's yield to maturity. For a bond with maturity at time $N$, the yield to maturity $\lambda(0, N)$ is defined such that

$$B(0, N) = \frac{1}{(1 + \lambda(0, N)\Delta T)^N}. \tag{3}$$

If we plot $B(0, N)$ as a function of $N$ it will be a decreasing graph. However, if we plot $\lambda(0, N)$ as a function of $N$ it will usually be an increasing graph. This reflects the fact that people usually require a higher compensation for a longer duration of the loan.

Furthermore, for a coupon paying bond with coupon rate $C$, face value $F$ and maturity $N$, its yield to maturity $\lambda(0, N)$ is defined such that

$$B(0, N) = \sum_{k=0}^{N} \frac{\frac{C}{100}F}{(1 + \lambda(0, N)\Delta T)^k} + \frac{F}{(1 + \lambda(0, N)\Delta T)^N}. \tag{4}$$

# 7 Forward rate agreement

A forward rate agreement (FRA) is a contract that allows the holder to pay a fixed rate $K(0, T_1, T_2)$ and receive a floating rate $L(T_1, T_2)$ for the loan period $[T_1, T_2]$. Suppose all interests are simple. The contract holder will receive (for a notional amount of 1 dollar)

$$(L(T_1, T_2) - K(0, T_1, T_2))(T_2 - T_1),$$

at time $T_2$. Here $L(T_1, T_2)$ is simply understood as the simple interest available for depositing or borrowing 1 dollar in the period $[T_1, T_2]$. A typical quote for $L(T_1, T_2)$ follows the Libor (London Interbank Offered Rate). Libor rates are calculated for 5 currencies and 7 borrowing periods ranging from overnight to one year and are published each business day by Thomson Reuters. For convenience, we can assume $L(T_1, T_2)$ falls into these 7 borrowing periods. If not, it can be figured out using certain interpolation methods from the given Libor rates. We emphasize that $L(T_1, T_2)$ is a random variable that is only known at time $T_1$ while $K(0, T_1, T_2)$ is a constant that is known at time $t = 0$.

Assume $T_1, T_2$ are two points on our discrete grid. The simple Libor rate is related to the compouding short rate as followed

$$1 + L(T_1, T_2)(T_2 - T_1) = \prod_{i=1}^{n}(1 + r_i(t_{i+1} - t_i)), \tag{5}$$

where we assume $t_1 = T_1$ and $t_n = T_2$.

The forward rate $f(0, T_1, T_2)$ is the fixed rate (the strike) $K(0, T_1, T_2)$ so that the FRA costs nothing to enter at the present time $t = 0$. It turns out that we can calculate $f(0, T_1, T_2)$ as followed:

$$f(0, T_1, T_2) = \frac{1}{T_2 - T_1}\left(\frac{B(0, T_1)}{B(0, T_2)} - 1\right).$$

The reasoning is as followed: suppose we enter a FRA at time 0 and have 1 dollar at time $T_1$. We can either do nothing with it, or invest it with the forward rate $f(0, T_1, T_2)$ to receive $(1 + f(0, T_1, T_2))(T_2 - T_1)$ at time $T_2$. These two schemes should have the same present value at time 0 to avoid arbitrage. That is

$$B(0, T_1) = B(0, T_2)(1 + f(0, T_1, T_2))(T_2 - T_1).$$

From this equation we can solve for $f(0, T_1, T_2)$ as indicated.

This equation also suggests a replication scheme for the FRA as followed: at time 0 we sell 1 FRA, short 1 share of zero coupon bond with maturity $T_1$ and long $(1 + f(0, T_1, T_2))(T_2 - T_1)$ shares of zero coupon bond with maturity $T_2$. At time $T_1$ we receive 1 dollar from the FRA buyer and we use it to close out our short position in the bond $B(0, T_1)$. At time $T_2$ we receive $(1 + f(0, T_1, T_2))(T_2 - T_1)$ from our long position with the bond $B(0, T_2)$ and use it to close out our short position in the FRA contract.

# 8 Interest rate futures

An interest rate future is a contract that costs nothing to enter, and allows the holder to pay a fixed rate $F(0, T_1, T_2)$ and receive a floating rate $R(T_1, T_2)$ for the loan period $[T_1, T_2]$. The contract holder is required, however, to post a margin account to reflect the fluctuation in the contract price as time moves from $t = 0$ to $t = T_1$.

In practice, the interest rate future is executed as followed:

1. The contract is settled at time $T_1$ and the contract holder does not (typically) enter into the loan period.

2. The contract holder is required to post a margin account. If the future rate $F(t, T_1, T_2)$ increases, he has to deposit more money into the account. If the future rate $F(t, T_1, T_2)$ decreases he receives money into the account.

3. The future contract is quoted with the price $100 - F(t, T_1, T_2)$ for $0 \leq t \leq T_1$. Its price at time $T_1$ is $100 - R(T_1, T_2)$.

If the interest rate is constant then the futures rate and forward rate are equal. However, in practice the forward rate is less than the future rate, due to the margining requirement. This phenomenon is called the negative convexity of the futures rate. Intuitively, it can be explained as followed. The margining always works against the futures contract holder (and works for the futures contract seller). When they receive margin, this will coincide with a drop in the rate at which the balance in the margin account is rolled. When they have to deposit margin, this will coincide with an increase in the borrowing rate. Thus if the forward rate is equal to the futures rate, one can buy a forward rate agreement and sell a futures contract to profit from the margining effect.

# 9 Duration and convexity of a bond

A bond price is a function of its convexity. See e.g. equations (3) and (4). Thus it is equivalent to quote either the bond price or the yield to maturity; and more often it is more important to look at the yield to maturity as an indication for the bond's "value". The yield to maturity is affected by other variables, such as the short rate (which may change by the FED's decision, for example). Thus we want to know how sensitive the bond price is with respect to a change in the yield to maturity. The concept that captures this is called the bond's duration:

$$
\begin{aligned}
D &= -\frac{1}{B(0,T)} \frac{dB(0,T)}{d\lambda(0,T)} \\
&= -\frac{d}{d\lambda(0,T)} \log(B(0,T)).
\end{aligned}
$$

The unit of the yield to maturity $\lambda(0,T)$ used in this formula is in percentage (for example yield going from 8 % per year ( $\lambda(0,T) = 0.08$ ) to 9 % per year ($\lambda(0,T) = 0.09$) ) . Thus the unit of duration $D$ is *percent change in price* (since it is $\frac{dB(0,T)}{B(0,T)}$ ) per one percentage point change in yield per year. The duration $D$ captures sensitivity of bond price to interest rate.

Remark: The definition we gave above technically is called the modified duration. There is another concept called the Macaulay duration which measures the weighted average time until cash flows are received. Let $C_i$ be a sequence of future cash flow that is received at time $t_i$, whose present value at time $t = 0$ is $P_i$. The present value of this cash flow is

$$V = \sum_i P_i.$$

The Macaulay duration is defined as

$$D_{Macaulay} = \sum_i \frac{P_i t_i}{V}.$$

It turns out that the Macaulay duration and the modified duration are pretty similar in practice. Thus we have the following important observation: the duration of a zero coupon bond is equal to its maturity (under continuous compounding) and the duration of a coupon paying bond is always less than its maturity. Generally speaking, the longer the duration of a bond, the riskier it is and the higher its price volatility is.

A related concept of duration is called the dollar duration. It is defined as

$$DV_{01} = -\frac{dB(0,T)}{10000 d\lambda(0,T)}.$$

Here $\lambda(0,T)$ is measured in the unit of basis point. One basis point equals .01 %, hence the 10000 at the denominator. Dollar duration or DV01 is the change in price in dollars, not in percentage. It gives the dollar variation in a bond's value per unit change (in basis point) in the yield.

A bond's convexity is defined as

$$C = \frac{1}{B(0,T)} \frac{d^2 dB(0,T)}{\lambda(0,T)^2}.$$

Thus one can approximate the change in the bond price with respect to the change in yield if one knows its (modified) duration and convexity as followed:

$$\frac{dB(0,T)}{B(0,T)} \approx -D_{mod} d\lambda(0,T) + \frac{1}{2} C (d\lambda(0,T))^2.$$

Given a portfolio consisting of $n$ bonds with present value $P_i, i = 1, \cdots, n$; modified durations $D_i, i = 1, \cdots, n$ and convexities $C_i, i = 1, \cdots, n$, we can calculate the duration and convexity of the portfolio as the weighted average of the individual convexity:

$$D_{portfolio} = \sum_{i=1}^{n} \frac{D_i P_i}{P},$$

where $P = \sum_{i=1}^{n} P_i$ and

$$C_{portfolio} = \sum_{i=1}^{n} \frac{C_i P_i}{P}.$$

# 10 Interest rate swaps

An interest rate swap, set over a period of times $t_0, t_1, \cdots, t_n$ is like a sequence of FRA where at each time $t_i$ one party (referred to as the payer) pays an agreed upon (fixed) rate in exchange for a floating rate, typically the Libor during the period $[t_i, t_{i+1}]$. The party who receives the fixed rate (and pays the floating rate) is referred to as the receiver. The contract is entered at time $t = 0$ and the first swap payment happens at $t = t_1$.

If we denote the fixed rate as $K$ then at each time $t_i$ the paying leg receives

$$(L(t_{i-1}, t_i) - K)(t_i - t_{i-1}).$$

We compute the present value of the total cash flow of the fixed leg and the floating leg separately. The present value of the total cash flow of the fixed leg is

$$\sum_{i=1}^{n} K(t_i - t_{i-1})B(0, t_i).$$

To compute the present value of the floating rate payment, we need to compute the present value $P_i$ of a payment of the type $L(t_{i-1}, t_i)(t_i - t_{i-1})$ received at time $t_i$. This is because $L(t_{i-1}, t_i)$ is not known until time $t_{i-1}$ and we need to use risk neutral valuation to compute its present value. Following the method we discuss aboved, we have

$$P_i = E^Q\Big(\prod_{k=1}^{i} \frac{1}{1 + r_k(t_k - t_{k-1})} L(t_{i-1}, t_i)(t_i - t_{i-1})\Big).$$

It turns out that we can express $P_i$ in a more informative way using our analysis with the FRA. Recall that $P_i$ is the price such that one is indifferent between receiving $P_i$ right now or receiving $L(t_{i-1}, t_i)(t_i - t_{i-1})$ at time $t_i$. If one to enter in to a FRA in the period $[t_{i-1}, t_i]$ with the forward rate $f(0, t_{i-1}, t_i)$, one would also be indifferent between receiving a fixed payment $f(0, t_{i-1}, t_i)(t_i - t_{i-1})$ or a floating payment $L(t_{i-1}, t_i)(t_i - t_{i-1})$. We know the present value of the fixed payment $f(0, t_{i-1}, t_i)(t_i - t_{i-1})$ is

$$\begin{aligned} B(0, t_i) f(0, t_{i-1}, t_i)(t_i - t_{i-1}) &= B(0, t_i) \frac{1}{t_i - t_{i-1}} \left(\frac{B(0, t_{i-1})}{B(0, t_i)} - 1\right)(t_i - t_{i-1}) \\ &= B(0, t_{i-1}) - B(0, t_i). \end{aligned}$$

This implies that

$$P_i = B(0, t_{i-1}) - B(0, t_i),$$

as well. Indeed, we can derive this result using the relation (5).

Thus the present value of the total cash flow for the floating payment leg is just

$$\sum_{i=1}^{n} P_i = 1 - B(0, t_n).$$

Similar to the forward rate, a swap rate $S_0$ is the fixed rate $H$ such that the swap contract costs nothing to enter. In other words, it is the fixed rate such that the present value of the fixed leg payment equals the present value of the floating leg payment. Using our above results, this means

$$S_0 \sum_{i=1}^{n} (t_i - t_{i-1}) B(0, t_i) = 1 - B(0, t_n).$$

Thus the swap rate is

$$S_0 = \frac{1 - B(0, t_n)}{\sum_{i=1}^{n} (t_i - t_{i-1}) B(0, t_i)}.$$

Remark: A swap contract has zero value at time 0 but most definitely will not at time $t_i, i > 0$. Indeed at time $t = 1$, a swap contract entered at time 0 will be similar to a $n - 1$ paying period contract entered at time 1. The difference is the rate has already been set for the contract entered at time 0 (namely $S_0$). On the other hand, if one enters a new contract at time $t = 1$, the swap rate will be

$$S_1 = \frac{1 - B(t_1, t_n)}{\sum_{i=2}^{n} (t_i - t_{i-1}) B(t_1, t_i)}.$$

It is easy to see that unless $S_1 = S_0$, the contract entered at time 0 will not have zero value at time 1. In particular, if $S_0 > S_1$ then the contract has positive value for the receiver side of the contract and negative value for the payer side of the contract.