

# The Greeks

Math 485

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## 1 Motivation:

Suppose an option seller sells a Euro-style derivative that pays  $V_T = \phi(S_T)$  at time  $T$ . We already learned that he should charge  $V_0 = E(e^{-rT}\phi(S_T))$  for the option at time 0.

Now the question is what should the option seller should do with  $V_0$ ? He is obligated to pay out  $\phi(S_T)$  (For example,  $\phi(S_T) = (S_T - K)^+$  if the derivative is a Euro Call option) at time  $T$ . Certainly he cannot just invest  $V_0$  in the bank and hope that he will have enough money to cover the random amount  $\phi(S_T)$  that needs to be paid out at time  $T$ . Clearly he needs to invest  $V_0$  in a portfolio that is a combination of the stock  $S$  and the money market.

But how much should he hold in stocks? Recall from the binomial tree model, we learned that to hedge a Euro-style derivative, at any time  $k$  the option seller should hold  $\Delta_k := \frac{V_{k+1} - V_k}{S_{k+1} - S_k}$  shares of stock and put the rest of his money into the money market. Then at the expiration time  $n$ , the value of his portfolio will be exactly equal to  $V_n$ , the amount that needs to be paid out. We will apply this idea in continuous time as well. This is the idea of Delta hedging.

## 2 Delta hedging:

The idea: We divide the interval  $[0, T]$  into  $n$  subintervals, each with length  $\delta$  ( $\delta$  small). We denote each grid point of these subintervals by  $t_k, 0 = t_0 < t_1 < \dots < t_n = T$ .

We construct a self-financing portfolio that consists of the underlying stock and the money market as followed: At each time  $t_k$ , we will hold  $\Delta_k := \frac{\partial V}{\partial S}(t_k)$  shares of stock.

We claim that in this way, the value of the portfolio at time  $T$  will approximately be equal to the value of the derivative  $V_T = \phi(S_T)$ .

Reason: By Ito's formula

$$V_{t_{k+1}} - V_{t_k} \approx \left( \frac{\partial V}{\partial t}(t_k) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t_k) \sigma^2 S_{t_k}^2 \right) \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k}).$$

Since  $V_t$  satisfies the Black-Scholes PDE, we have

$$\frac{\partial V}{\partial t}(t_k) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t_k) \sigma^2 S_{t_k}^2 = -\frac{\partial V}{\partial S} r S(t_k) + r V(t_k).$$

Plug this in the above:

$$\begin{aligned} V_{t_{k+1}} - V_{t_k} &\approx \left( -\frac{\partial V}{\partial S} r S(t_k) + r V(t_k) \right) \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k}) \\ &= \left( V(t_k) - \frac{\partial V}{\partial S} S(t_k) \right) r \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k}). \end{aligned}$$

Now suppose at time  $t_k$  we have a portfolio  $\pi$  that satisfies  $\pi(t_k) \approx V(t_k)$ . We purchase  $\frac{\partial V}{\partial S}(t_k)$  shares of stock, which leaves us with  $\pi(t_k) - \frac{\partial V}{\partial S} S(t_k)$  to put into the bank. At time  $t_{k+1}$  the value of our portfolio is (because of self-financing)

$$\pi(t_{k+1}) = \pi(t_k) + \left( \pi(t_k) - \frac{\partial V}{\partial S} S(t_k) \right) r \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k})$$

Note that we're in discrete time so the growth in 1 period of time of the money market portion is the interest rate times the length of that period, which is  $\delta$ .

But since  $\pi(t_k) \approx V(t_k)$  we have

$$\begin{aligned} \pi(t_{k+1}) &\approx V(t_k) + \left( V(t_k) - \frac{\partial V}{\partial S} S(t_k) \right) r \delta + \frac{\partial V}{\partial S}(t_k) (S_{t_{k+1}} - S_{t_k}) \\ &\approx V(t_{k+1}). \end{aligned}$$

So the approximation extends to the next period. The quantity  $\frac{\partial V}{\partial S}$ , the first partial derivative of  $V$  with respect to  $S$ , is thus seen to be very important in hedging, and it's called the Delta, in symbol  $\Delta$ , the first Greek we encounter in this section.

### 3 Computing $\frac{\partial V}{\partial S}$ :

The above derivation is valid for any Euro-style derivative. However, the relevant question is: how much is exactly  $\frac{\partial V}{\partial S}$ ? Or how to compute the Delta of a certain Euro derivative? This is difficult in general and usually one needs to use numerical

techniques. However, when we specialize to certain cases of  $V_T = \phi(S_T)$ , for example  $\phi(S_T) = S_T^k$  for some integer  $k$  then explicit computation of the Delta is possible. In this section we show how to compute the Delta of the most important derivative we encounter in this class: the Euro-Call option.

Recall that the Black-Scholes formula gives for a Euro call that pays  $(S_T - K)^+$  at time  $T$ :

$$\begin{aligned} V(t, S_t) &= S_t N(d_1(t, S_t)) - K e^{-r(T-t)} N(d_2(t, S_t)), \\ d_1(t, S_t) &= \frac{(r + \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}} \\ d_2(t, S_t) &= \frac{(r - \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}} \end{aligned}$$

It is also easy to see that

$$\frac{\partial}{\partial S} d_2(t, S_t) = \frac{\partial}{\partial S} d_1(t, S_t) = \frac{1}{S_t \sigma \sqrt{T-t}}.$$

Therefore,

$$\frac{\partial V}{\partial S}(t) = N(d_1(t, S_t)) + S_t \phi_Z(d_1(t, S_t)) \frac{1}{S_t \sigma \sqrt{T-t}} - K e^{-r(T-t)} \phi_Z(d_2(t, S_t)) \frac{1}{S_t \sigma \sqrt{T-t}}.$$

We claim that

$$\phi_Z(d_1(t, S_t)) = K e^{-r(T-t)} \phi_Z(d_2(t, S_t)) \frac{1}{S_t}.$$

To see this, note that  $d_1(t, S_t) = d_2(t, S_t) + \sigma\sqrt{T-t}$ . Therefore,

$$\begin{aligned} \phi_Z(d_1) &= \phi_Z(d_2 + \sigma\sqrt{T-t}) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_2 + \sigma\sqrt{T-t})^2}{2}\right) \\ &= \phi_Z(d_2) \exp\left(\frac{-2d_2\sigma\sqrt{T-t} - \sigma^2(T-t)}{2}\right). \end{aligned}$$

One can check that

$$2d_2(t, S_t)\sigma\sqrt{T-t} + \sigma^2(T-t) = 2(r(T-t) - \log(K) + \log(S_t)).$$

Plug this into the above expression, the claim is checked. Thus we see a surprisingly simple result:  $\frac{\partial V}{\partial S}(t) = N(d_1(t, S_t))$ .

## 4 Predicting the future price of Euro Call option - Theta, Delta and Gamma

So we see that the partial derivative of  $V$  with respect to  $S$  plays an important role in hedging. Indeed the Greeks are just various partial derivatives of  $V$  with respect to different parameters in the Black-Scholes model:  $t, r, \sigma, T, S$ . Some of them show up more often than others. In particular, two more Greeks that are important for our purpose are the ones that appear in Ito's formula:

$$\begin{aligned}\Theta(t) &:= \frac{\partial V}{\partial t}(t) \\ \Gamma(t) &:= \frac{\partial^2 V}{\partial S^2}(t),\end{aligned}$$

and of course previously we have

$$\Delta(t) := \frac{\partial V}{\partial S}(t).$$

Note that in this way the Greeks are random processes. They are functions of  $t$  and  $S_t$ . Their use is to measure the sensitivity of the option price with respect to the change of other parameters in the model. Again in general it may be difficult to compute the  $\Theta, \Gamma$  of a general derivative. But if we specialize to certain form of  $\phi(S_T)$  then the computation can be doable. In particular, for the Euro-Call option:

$$\begin{aligned}\Gamma(t) &= \frac{1}{\sigma S_t \sqrt{2\pi(T-t)}} e^{-\frac{d_1^2(t, S_t)}{2}} \\ \Theta(t) &= -r e^{-r(T-t)} K N(d_2(t, S_t)) - \frac{1}{2} \sigma^2 S_t^2 \Gamma(t).\end{aligned}$$

The formulas are complicated, but they are explicit and one can compute these quantities provided  $S_t, \sigma, r, T$  are given. Also at time  $t$ , using Black-Scholes formula we also know  $V_t$ . Therefore, Ito's formula gives for a small change in time  $t + \delta$

$$V_{t+\delta} \approx V_t + (\Theta(t) + \frac{1}{2} \Gamma(t) \sigma^2 S^2(t)) \delta + \Delta(t) (S_{t+\delta} - S_t).$$

Note: The book used  $V_{new}$  for  $V_{t+\delta}$  and only consider the case  $t = 0$ . So their formula is simpler than ours and our formula is slightly more general.

## 5 Comparing option price - Vega and Rho

### 5.1 Vega - $\nu(t, S_t)$

We're interested in the following question: Suppose

$$\begin{aligned}dS_t^i &= \mu^i S_t^i + \sigma S_t^i dW_t, \\S_0^i &= S_0,\end{aligned}$$

under the physical measure  $P$ . That is the two stocks have different average return and same volatility, with the same initial price. Let  $V_t^i$  be the price of Euro Call option with expiration  $T$  and strike  $K$  on  $S^i$ . Suppose that  $\mu^1 > \mu^2$ . Can we conclude that  $V_t^1 > V_t^2$ ?

This is actually a trick question. The answer is NO,  $V_t^1 = V_t^2$ . The reason is we need to price these options under the risk neutral measure. And under the risk neutral measure, the dynamics of  $S^i$  are

$$\begin{aligned}dS_t^i &= rS_t^i + \sigma S_t^i dW_t, \\S_0^i &= S_0.\end{aligned}$$

That is they have the SAME dynamics with the same initial condition. So the corresponding call options on them have the same price.

A more interesting question would be what if they have different volatility? Suppose that

$$\begin{aligned}dS_t^i &= rS_t^i + \sigma^i S_t^i dW_t, \\S_0^i &= S_0,\end{aligned}$$

where  $\sigma^1 > \sigma^2$  under  $Q$ . What can we say about  $V_t^1$  versus  $V_t^2$ ? Note that appealing to the Black-Scholes formula is not straightforward because

$$\begin{aligned}V(t, S_t) &= S_t N(d_1(t, S_t)) - Ke^{-r(T-t)} N(d_2(t, S_t)), \\d_1(t, S_t) &= \frac{(r + \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}}, \\d_2(t, S_t) &= \frac{(r - \frac{1}{2}\sigma^2)(T-t) - \log(\frac{K}{S_t})}{\sigma\sqrt{T-t}},\end{aligned}$$

and we have the presence of  $\sigma$  at BOTH the numerator and the denominator of the fractions in  $d_1, d_2$ . So one can't say straightforwardly that if  $\sigma$  increases then  $V(t, S_t)$  increases.

The way to answer this question is to differentiate  $V(t, S_t)$  with respect to  $\sigma$  and look at the sign of the derivative. Doing this is equivalent to find the Greek  $\nu(t, S_t) = \frac{\partial}{\partial \sigma} V(t, S_t)$  of the Euro Call option. The computation is as followed: note that

$$\begin{aligned}\frac{\partial}{\partial \sigma} d_1(t, S_t) &= \frac{-r + \log(\frac{K}{S_t})}{\sigma^2 \sqrt{T-t}} + \frac{1}{2} \sqrt{T-t} \\ \frac{\partial}{\partial \sigma} d_2(t, S_t) &= \frac{-r + \log(\frac{K}{S_t})}{\sigma^2 \sqrt{T-t}} - \frac{1}{2} \sqrt{T-t}.\end{aligned}$$

Therefore

$$\begin{aligned}\nu(t, S_t) &= S_t \phi_Z(d_1(t, S_t)) \left[ \frac{-r + \log(\frac{K}{S_t})}{\sigma^2 \sqrt{T-t}} + \frac{1}{2} \sqrt{T-t} \right] \\ &\quad - K e^{-r(T-t)} \phi_Z(d_2(t, S_t)) \left[ \frac{-r + \log(\frac{K}{S_t})}{\sigma^2 \sqrt{T-t}} - \frac{1}{2} \sqrt{T-t} \right].\end{aligned}$$

This, coupled with the fact discussed above that

$$\phi_Z(d_1(t, S_t)) = K e^{-r(T-t)} \phi_Z(d_2(t, S_t)) \frac{1}{S_t}$$

gives a rather simple formula for  $\nu$ :

$$\nu(t, S_t) = S_t \phi_Z(d_1(t, S_t)) \sqrt{T-t}.$$

We can now answer our original question. If  $\sigma_1 > \sigma_2$  then  $V_t^1 > V_t^2$ , since  $\nu(t, S_t) \geq 0$ .

Remark: The above result can also be justified with the following intuitive argument: since a call option is like an insurance, and since the larger the volatility, the riskier the stock is, the call option on the stock with the higher volatility should have a higher price. Note that this argument also applies to the put option. So we predict that the put option on the stock with the higher volatility should have a higher price. We do NOT have to compute the vega of the Put option to justify this. It simply follows from the Put-Call parity:  $V_t^c - V_t^p = S_t - K$ . Thus if  $V_t^c$  increases,  $V_t^p$  also have to increase.

## 5.2 Rho - $\rho(t, S_t)$

We consider the following situation

$$\begin{aligned}dS_t^i &= r^i S_t^i + \sigma S_t^i dW_t, \\S_0^i &= S_0,\end{aligned}$$

and  $V_t^i$  corresponding price of Euro Call option on  $S^i$  with strike  $K$  and expiration  $T$ . This can be interpreted as evaluating the price of the call option on the same stock in different periods of the economy that has different interest rate. Can we compare  $V^1$  and  $V^2$ , say if  $r^1 > r^2$ ?

From the discussion in the previous section, you see that we need to compute  $\rho(t, S_t) = \frac{\partial}{\partial r} V(t, S_t)$ . You can verify that

$$\rho(t, S_t) = K(T-t)e^{-r(T-t)}N(d_2(t, S_t)).$$

Thus, since  $\rho(t, S_t) \geq 0$ , we also have  $V_t^1 > V_t^2$ .