Math 251
Fall 2015
Midterm 2
11/17/15

Name (Print): ______________________

This exam contains 4 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Compute

$$\iint_{\mathcal{R}} xy \;\; dxdy$$

where $\mathcal{R}$ is the circle centered at $(2,2)$ with radius 2.

Ans:

$$\iint_{\mathcal{R}} xy \;\; dxdy = \iint_{\mathcal{R}'} (x+2)(y+2)dxdy$$

where $\mathcal{R}'$ is the circle centered at 0 with radius 2. Using polar coordinate

$$\begin{aligned} \iint_{\mathcal{R}'} (x+2)(y+2)dxdy &= \int_0^{2\pi}\int_0^2 (r\cos\theta+2)(r\sin\theta+2)rdrd\theta \\ &= \int_0^{2\pi}\int_0^2 \left[r^2\cos\theta\sin\theta + 2r(\cos\theta+\sin\theta)+4)\right]rdrd\theta. \end{aligned}$$

Computing each term in the sum:

$$\begin{aligned} \int_0^{2\pi}\int_0^2 r^3\cos\theta\sin\theta drd\theta &= \left(\int_0^2 r^3dr\right)\left(\int_0^{2\pi}\cos\theta\sin\theta d\theta\right) \\ &= 4\times 0 = 0 \\ \int_0^{2\pi}\int_0^2 2r^2(\cos\theta+\sin\theta)drd\theta &= \left(\int_0^2 2r^2dr\right)\left(\int_0^{2\pi}(\cos\theta+\sin\theta)d\theta\right) \\ &= 16/3\times 0 = 0 \\ \int_0^{2\pi}\int_0^2 4rdrd\theta &= \left(\int_0^2 4rdr\right)\left(\int_0^{2\pi} d\theta\right) = 8\times 2\pi = 16\pi. \end{aligned}$$

Thus the answer is 16π.

2. (20 points) Find the point on the ellipse $5x^2+5y^2-6xy=8$ that has the largest x coordinate.

Ans: We want to max x subject to $5x^2+5y^2-6xy=8$. The Lagrange multiplier condition is

$$\langle 1,0\rangle = \lambda\langle 10x-6y, 10y-6x\rangle.$$

We conclude that $5y=3x$ and $\lambda(10x-6y)=1$. The first equality implies

$$(5+9/5-18/5)x^2 = 8$$

or $x=\pm\sqrt{\frac{5}{2}}$. Thus $\sqrt{\frac{5}{2}}$ is the largest x coordinate on the ellipse.

3. (20 points) Let $\mathcal{R}$ be the region bounded by the parallelogram that goes through the points $(0,0),(2,0),(3,1),(1,1)$. Compute the double integral

$$\iint_{\mathcal{R}} xy \;\; dxdy.$$

Ans: The region can be described by the points (x,y) such that

$$\begin{aligned} 0\le x-y &\le 2 \\ 0\le y &\le 1. \end{aligned}$$

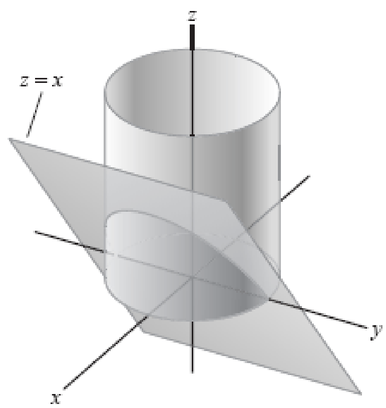
Leting $u = x - y$ and $v = y$ leads to the change of variables:

$$\iint_{\mathcal{R}} xy \;\; dxdy = \int_0^1 \int_0^2 (u+v)v \;\; dudv.$$

We evaluate

$$\int_0^1 \int_0^2 (u+v)v \;\; dudv = (\int_0^2 udu)(\int_0^1 vdv) + 2\int_0^1 v^2 dv = 5/3.$$

4. (20 points) Find the volume of the wedge shaped region as in the picture below contained in the cylinder $x^2 + y^2 = 9$, bounded above by the plane $z = x$ and below by the xy plane.



Ans: Using cylindrical coordinate, the region is bounded by the surfaces $\{z = r\cos\theta\}, \{z = 0\}, \{-\pi/2 \le \theta \le \pi/2, r = 3\}$. Thus the volume of the region is

$$\int_{-\pi/2}^{\pi/2} \int_0^3 \int_0^{r\cos\theta} rdzdrd\theta = (\int_0^3 r^2 dr)(\int_{-\pi/2}^{\pi/2} \cos\theta d\theta) = 18.$$

5. (20 points) Let $\mathbf{F}$ be a vector field of the form

$$\mathbf{F} = \langle \frac{z}{x}, \frac{z}{y}, \log(xy) \rangle.$$

Let C be the oriented curve that is determined by the helix

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle, \frac{\pi}{6} \le t \le \frac{\pi}{3}.$$

Compute the path integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

(Hint: Investigate to see if $\mathbf{F}$ has any special property. You can also use the fact that $\cos(\pi/6) = \sin(\pi/3) = \frac{\sqrt{3}}{2}, \cos(\pi/3) = \sin(\pi/6) = 1/2$.)

Ans: $\mathbf{F}$ has a potential function $V(x, y, z) = z\log(xy)$. V is undefined at $(x, y) = (0, 0)$. However, the curve given entirely lies in the positive (x, y, z) region so $\mathbf{F}$ is conservative in a domain containing this curve. Thus applying the fundamental theorem for line integral we have

$$\int_C \mathbf{F} \cdot d\mathbf{s} = V(\mathbf{r}(\pi/3)) - V(\mathbf{r}(\pi/6)) = \pi/3 \log(\sqrt{3}/4) - \pi/6 \log(\sqrt{3}/4) = \pi/6 \log(\sqrt{3}/4).$$