Math 21
Fall 2015
Midterm 1
10/13/2015

Name (Print): ______________________

This exam contains 5 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	15	
4	15	
5	10	
6	20	
Total:	100	

1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in $\mathbb{R}^3$ such that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle 0, 1, 1 \rangle \\ \mathbf{v} \times \mathbf{w} &= \langle 1, 0, 1 \rangle. \end{aligned}$$

We also know that $\|\mathbf{v}\| = 1$ and $\mathbf{v} \cdot \langle 1, 0, 0 \rangle > 0$.

(a) (15 points) Solve for $\mathbf{v}$.

Ans: Observe that $\mathbf{v}$ is orthogonal to both $\langle 0, 1, 1 \rangle$ and $\langle 1, 0, 1 \rangle$. Thus $\mathbf{v}$ is parallel to their cross product, which is $\langle 1, 1, -1 \rangle$. Now the restrictions $\|\mathbf{v}\| = 1$ and $v \cdot \langle 1, 0, 0 \rangle > 0$ give $\mathbf{v} = \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle$.

(b) (5 points) Solve for one example of $\mathbf{u}$ (there may not be a unique $\mathbf{u}$ so that $\mathbf{u} \times \mathbf{v} = \langle 0, 1, 1 \rangle$. You only need to find one example that works here.)

Ans: This is one possible solution: we know that $\mathbf{u}$ is orthogonal to $\langle 0, 1, 1 \rangle$ and

$$\|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta) = \|\langle 0, 1, 1 \rangle\| = \sqrt{2}.$$

If we also impose the angle θ between $\mathbf{u}$ and $\mathbf{v}$ to be $\pi/2$ then the above reads $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\langle 0, 1, 1 \rangle$,

$$\|\mathbf{u}\| = \sqrt{2}.$$

This suggests that $\mathbf{u}$ is parallel to the cross product of $\langle 1, 1, -1 \rangle$ and $\langle 0, 1, 1 \rangle$, which is $\langle 2, -1, 1 \rangle$. We need to make sure $\|\mathbf{u}\| = \sqrt{2}$, thus

$$\mathbf{u} = \pm \frac{1}{\sqrt{3}} \langle 2, -1, 1 \rangle.$$

Lastly we just need to plug in :

$$\frac{1}{\sqrt{3}} \langle 2, -1, 1 \rangle \times \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle = \langle 0, 1, 1 \rangle,$$

so indeed one example of $\mathbf{u}$ is $\frac{1}{\sqrt{3}} \langle 2, -1, 1 \rangle$.

2. Let $\mathcal{P}$ be the plane $2x + y + z = 2$ and S be the sphere $x^2 + y^2 + z^2 = 2$.

   (a) (5 points) Let $\mathbf{u}$ be the *unit* normal vector of $\mathcal{P}$. What is $\mathbf{u}$?
   Ans: $\frac{1}{\sqrt{6}}\langle 2, 1, 1\rangle$.

   (b) (10 points) Find two more vectors $\mathbf{v}, \mathbf{w}$ so that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthonormal set. That is we require they are all unit vectors and

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = \mathbf{0}.$$

   Ans: We can choose $\mathbf{v} = \frac{1}{\sqrt{2}}\langle 0, 1, -1\rangle$. Then

$$\mathbf{w} = \frac{\mathbf{v} \times \mathbf{u}}{\|\mathbf{v} \times \mathbf{u}\|} = \frac{1}{\sqrt{3}}\langle -1, 1, 1\rangle.$$

   (c) (5 points) Find the radius of the circle that is the intersection between $\mathcal{P}$ and S (Hint: consider the parametrization of this circle using the coordinate system associated with $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$).
   Ans: We will find the parametrization in terms of the coordinate system $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Let $\mathbf{r}(t)$ be a point on the curve. Then

$$\begin{aligned} \|\mathbf{r}(t)\|^2 &= 2 \\ \mathbf{u} \cdot \mathbf{r}(t) &= \frac{2}{\sqrt{6}}. \end{aligned}$$

   Now if we represent $\mathbf{r}(t)$ in terms of the new coordinate sytem it reads

$$\mathbf{r}(t) = (\mathbf{r}(t) \cdot \mathbf{u})\mathbf{u} + (\mathbf{r}(t) \cdot \mathbf{v})\mathbf{v} + (\mathbf{r}(t) \cdot \mathbf{w})\mathbf{w}.$$

   Therefore

$$\|\mathbf{r}(t)\|^2 = \frac{4}{6} + (\mathbf{r}(t) \cdot \mathbf{v})^2 + (\mathbf{r}(t) \cdot \mathbf{w})^2 = 2.$$

   Hence

$$(\mathbf{r}(t) \cdot \mathbf{v})^2 + (\mathbf{r}(t) \cdot \mathbf{w})^2 = 4/3.$$

   From this we conclude that the radius of the circle is $\sqrt{\frac{4}{3}}$.

3. Consider the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, a, b > 0$.
   (a) (10 points) Find a parametric representation of the ellipse in terms of $\cos t, \sin t$.
   Ans: $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$.
   (b) (5 points) Find the curvature $\kappa(t)$ of the curve at time t. (Hint: the formula

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

   may be useful).

   Ans: We have

$$\begin{aligned} \mathbf{r}'(t) &= \big\langle -a\sin t, b\cos t\big\rangle \\ \mathbf{r}''(t) &= \big\langle -a\cos t, -b\sin t\big\rangle, \end{aligned}$$

   Embedding these two vectors in 3d (so that we can take the cross product) reads

$$\begin{aligned} \mathbf{r}'(t) &= \big\langle -a\sin t, b\cos t, 0\big\rangle \\ \mathbf{r}''(t) &= \big\langle -a\cos t, -b\sin t, 0\big\rangle. \end{aligned}$$

   Thus

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \big\langle 0, 0, ab\big\rangle.$$

   Thus

$$\kappa(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}.$$

4. (15 points) Consider the function

$$\begin{aligned} f(x,y) &= \frac{x^2 - y^2}{x^2 + y^2}, (x,y) \neq 0 \\ &= 0, \quad (x,y) = 0. \end{aligned}$$

   Is $f(x,y)$ continuous at $(0,0)$?

   Ans: Fix $x = 0$

$$\lim_{y\to 0} \frac{x^2 - y^2}{x^2 + y^2} = -1.$$

   Fix $y = 0$

$$\lim_{x\to 0} \frac{x^2 - y^2}{x^2 + y^2} = 1.$$

   Therefore the limit as $(x,y) \to (0,0)$ of $f(x,y)$ does not exist. Hence it is not continuous at $(0,0)$.

5. Consider the function $f(x,y) = x^3y^2$.

(a) (5 points) Find the equation for the tangent plane to the surface $\langle x, y, f(x,y)\rangle$ at $(x,y) = (2,1)$.
Ans: $f_x(2,1) = 3 \times 2^2 \times 1^2 = 12, f_y(2,1) = 2 \times 2^3 \times 1 = 16$. Thus the equation of the tangent plane is

$$z = 8 + 12(x-2) + 16(y-1).$$

(b) (5 points) Find the linear approximation for $(2.01)^3(1.02)^2$.

Ans: We have

$$(2.01)^3(1.02)^2 \approx 2^3 \times 1^2 + 12 \times 0.01 + 16 \times 0.02 = 8 + 0.12 + 0.32 = 8.44.$$

6. (20 points) Consider the function $f(x,y) = x^3 - 2y$. Find the global extreme values of $f(x,y)$ over the rectangle $0 \le x \le 1, 0 \le y \le 1$.

Ans: $f_x = 3x^2, f_y = -2$ so there are no critical points. We look for the extreme values on the boundary. Since $f(x,y)$ is decreasing in y, the maximum happens on $y = 0$ and minimum happens on $y = 1$. Since f is increasing in x, the maximum happens at $(1,0)$ where $f(1,0) = 1$ and minimum happens at $(0,1)$ where $f(0,1) = -2$.