Math 251
Fall 2015
Final exam
12/17/15

Name (Print): ____________________

This exam contains 5 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total:	200	

1. A vector field $\mathbf{F}$ is said to have a vector potential if there is a vector field $\mathbf{A}$ so that $\mathbf{F} = curl(\mathbf{A})$. For the following vector fields, verify whether or not $\mathbf{F}$ has a vector potential. If not explain why, if yes find $\mathbf{A}$.

   (a) (10 points) $\mathbf{F} = \langle 2y-1, 3z^2, 2xy \rangle$.
   Note that $div(\mathbf{F}) = 0$ here so $\mathbf{F}$ passes the necessary condition test. We need to find $\mathbf{A}$ such that

$$\begin{aligned}(V_3)_y - (V_2)_z &= 2y-1\\ (V_1)_z - (V_3)_x &= 3z^2\\ (V_2)_x - (V_1)_y &= 2xy.\end{aligned}$$

   A possible choice is $V_2 = 0$, which leads to

$$\begin{aligned}V_1 &= -xy^2\\ V_2 &= 0\\ V_3 &= y^2 - y - 3z^2 x.\end{aligned}$$

   (b) (10 points) $\mathbf{F} = \langle x, -2y+z, 2z+3x \rangle$.
   Ans: $div(\mathbf{F}) = 1 - 2 + 2 = 1 \neq 0$ so there exists no such $\mathbf{A}$.

2. Let V be a region in 3-d enclosed by a surface S. We do not know the shape of V.

   (a) (10 points) Suppose we know additionally that

$$\iint_S \langle x + xy + z, x + 3y - \frac{1}{2}y^2, 4z \rangle \cdot d\mathbf{S} = 16.$$

   Can we find the volume of V? If yes find it, if no explain.
   Ans: Since $div\langle x + xy + z, x + 3y - \frac{1}{2}y^2, 4z\rangle = 1 + y + 3 - y + 4 = 8$ we can use divergence theorem to find the volume of V. Indeed it follows that $8V = 16$ or $V = 2$.

   (b) (10 points) Suppose we know instead that

$$\iint_S \langle x + xy + z, x + 3y - \frac{1}{2}y, 4z \rangle \cdot d\mathbf{S} = 16.$$

   Can we find the volume of V? If yes find it, if no explain.
   Ans: Here $div\langle x + xy + z, x + 3y - \frac{1}{2}y, 4z\rangle = 1 + y + 3 - 1/2y + 4 = 8 + 1/2y$, so we cannot evaluate the triple integral $\iiint_V (8+1/2y)dxdydz$ without knowing additional details about V.

3. (20 points) Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   where $a, b > 0$. Find the area of this ellipse using a method that we covered in class (simply quoting a known formula will not earn credit here).

   Ans: Let C denote the ellipse curve. By Stoke's theorem the area of the ellipse is $\int_C xdy$. A parametrization of this ellipse is $\langle a\cos(\theta), b\sin(\theta)\rangle, 0 \le \theta \le 2\pi$. Thus

$$\int_C xdy = \int_0^{2\pi} a\cos(\theta)b\cos(\theta)d\theta = \pi ab.$$

4. (20 points) Find the global max and min of $f(x,y) = (x^2-y^2)e^{x^2+y^2}$ over the domain $x^2+y^2 \le 2$.

   Ans: Using polar coordinate we can write $f(x,y) = f(r,\theta) = r^2\cos(2\theta)e^{r^2}$ and the domain is $0 \le \theta \le 2\pi, 0 \le r \le \sqrt{2}$. Note that $cos(2\theta)$ has a maximum of 1 at $\theta = 0, \pi$ and minimum of -1 at $\theta = \pi/2, 3\pi/2$. Thus the global max happens at $\theta = 0, \pi, r = \sqrt{2}$ with value $2e^2$ and the global min happens at $\theta = \pi/2, 3\pi/2$ with value $-2e^2$.

5. (20 points) The plane
$$\frac{x}{2} + \frac{y}{4} + \frac{z}{3} = 1$$
   intersects the x, y, z axes in points A, B and C. Find the area of the triangle ABC.

   Ans: $A = (2,0,0), B = (0,4,0), C = (0,0,3)$ and thus $AB = \langle -2,4,0\rangle, AC = \langle -2,0,3\rangle$. We can compute $AB \times AC = \langle 12,6,8\rangle$. The area of the triangle is
$$\frac{1}{2}||AB \times AC|| = \frac{1}{2}\sqrt{144+36+64} = \frac{1}{2}\sqrt{244}.$$

6. (20 points) Consider the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
   where $a, b > 0$. Find its perimeter using a method that we covered in class (simply quoting a known formula will not earn credit here).

   Ans: Using the parametrization $r(\theta) = \langle a\cos(\theta), b\sin(\theta)\rangle, 0 \le \theta \le 2\pi$ the perimeter P is
$$\begin{aligned} P &= \int_0^{2\pi} ||r'(\theta)||d\theta \\ &= \int_0^{2\pi} \sqrt{a^2\sin^2(\theta) + b^2\cos^2(\theta)}d\theta. \end{aligned}$$

   This integral has no closed form formula and this integral is the best we can do for the perimeter of the ellipse.

7. (20 points) Let R be the parallelogram region defined by four points $(0,0), (5,1), (3,4), (8,5)$. Evaluate the double integral
$$\iint_R xydxdy.$$

   Ans:

   We want to transform the integration to the unit square region $0 \le u \le 1, 0 \le v \le 1$. We seek a, b, c, d so that
$$\begin{aligned} x &= au + bv \\ y &= cu + dv, \end{aligned}$$

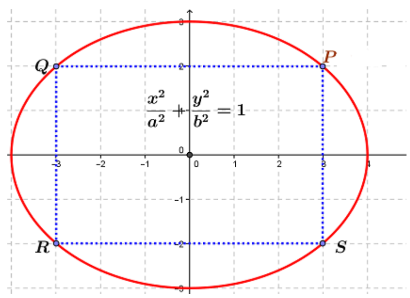
where the correspondence is $(0,0) \to (0,0), (0,1) \to (3,4), (1,0) \to (5,1), (1,1) \to (8,5)$. We have

$$\begin{aligned} x &= 5u + 3v \\ y &= u + 4v. \end{aligned}$$

The determinant of the Jacobian is $5 \times 4 - 3 \times 1 = 17$ and the integral becomes

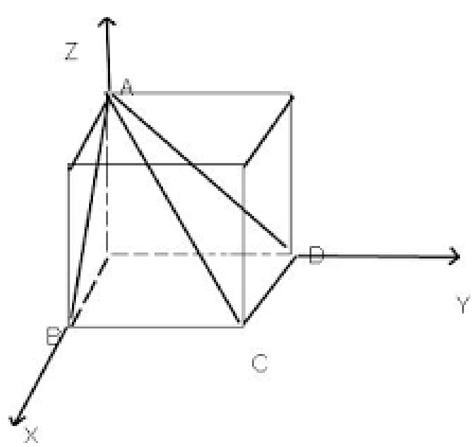
$$17 \int_0^1 \int_0^1 5u^2 + 23uv12v^2 = 17(5/3 + 23/4 + 4).$$

8. (20 points) Consider an ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
where $a, b > 0$. Find the maximum area of a rectangle inscribed in this ellipse (see picture for a case of such a rectangle).



Ans: We want to max $4xy$ subject to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is equivalent to max $4\frac{ab}{2}\sin(2\theta)$ subject to $0 \le \theta \le 2\pi$. Since $ab > 0$ this value is max when $\sin(2\theta) = 1$ or $\theta = \pi/4$. In this case the area of the rectangle is $2ab$.

9. (20 points) Consider the unit cube in 3-d as given in the following picture (so that each side of the cube has length one). Compute the angle $\widehat{\text{BAC}}$ between AB and AC, as well as the angle $\widehat{\text{BAD}}$ between AB and AD. If we also let $\widehat{\text{CAD}}$ as the angle between AC and AD is it true that $\widehat{\text{BAC}} + \widehat{\text{CAD}} = \widehat{\text{BAD}}$? Explain.



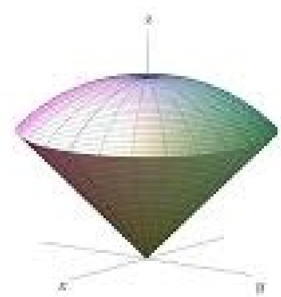
Ans: We have $A = (0,0,1), B = (1,0,0), C = (1,1,0), D = (0,1,0)$. Thus $AB = \langle 1, 0, -1 \rangle, AC =$

$\langle 1,1,-1\rangle, AD = \langle 0,1,-1\rangle$. Using the cosine formula in dot product, we have

$$\cos(\widehat{\mathrm{BAC}}) = \frac{AB\cdot AC}{||AB||||AC||} = \frac{2}{\sqrt{6}}$$
$$\cos(\widehat{\mathrm{BAD}}) = \frac{AB\cdot AD}{||AB||||AD||} = \frac{1}{2}.$$

This allows us to compute the angles $\widehat{\mathrm{BAC}}$ and $\widehat{\mathrm{BAD}}$. In particular we know that $\widehat{\mathrm{BAD}} = \frac{\pi}{3}$. It is not true that $\widehat{\mathrm{BAC}} + \widehat{\mathrm{CAD}} = \widehat{\mathrm{BAD}}$ since C does not lie in the same plane as A, B, D.

10. (20 points) Find the volume of the icecream cone region that lies below the sphere $x^2+y^2+z^2 = 1$ and the cone $x^2+y^2 = z^2$.



Ans: Using spherical coordinate:

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin(\phi)d\rho d\phi d\theta = \frac{(1-\frac{\sqrt{2}}{2})2\pi}{3}.$$