Differentiation of functions from $\mathbb{R}^n$ to $\mathbb{R}$

Math 251

October 1, 2015

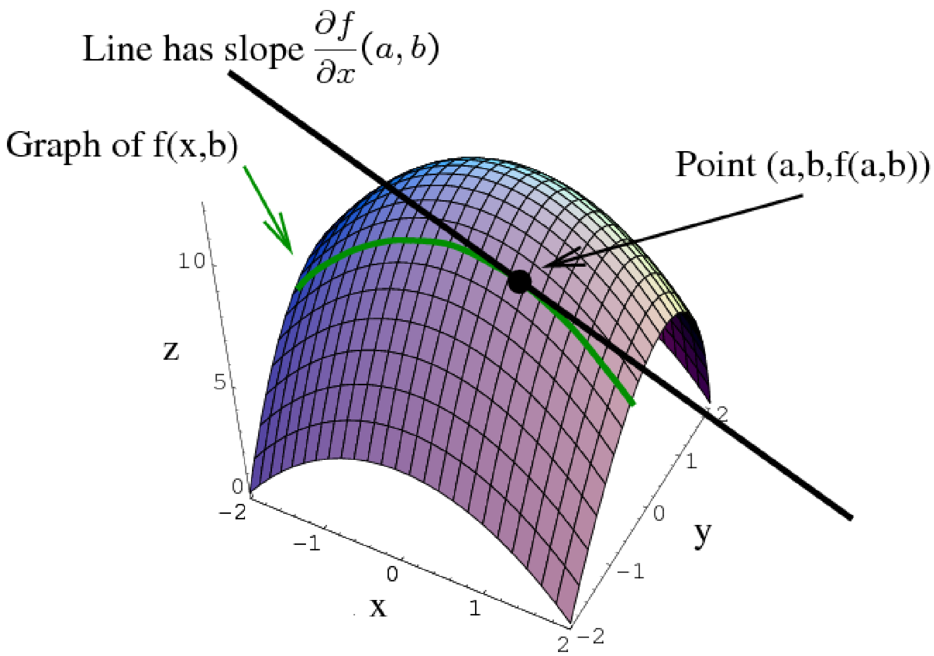
1 Partial derivatives

Consider $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$. f is naturally a function of two variables. However, if we fixed one of the variable, for example set $y = 0$, then $f(x,0)$ is a single-variable function that we have studied in Calculus 1. Then it makes sense to discuss the derivative in x of the function $f(x,0)$, at a point $x = a$ namely

$$\frac{d}{dx} f(a,0) = \lim_{h \to 0} \frac{f(a+h,0) - f(a,0)}{h}.$$

But you see there is nothing special about the point 0 in the above analysis. In general, we can fix y around any value c and discuss the derivative in x of the single variable function $f(x,c)$. Then we observe that we can drop the letter c altogether and just remember that when we differentiate in x we fix the variable y. Thus we arrive at the notion of the partial derivative of $f(x,y)$ with respect to x at a point $(x,y) = (a,b)$:

$$\frac{\partial}{\partial x} f(a,b) = f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$



Similarly one can define the partial derivative of $f(x,y)$ with respect to y:

$$\frac{\partial}{\partial y} f(a,b) = f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Observe that f_x is still a multi-variate function in (x,y). For example if

$$f(x,y) = x + y^2$$

then $f_x(x,y) = 1$.

Note that f_x and f_y are two different functions. For example in the above if

$$f(x,y) = x + y^2$$

then $f_x(x,y) = 2y$.

Because the first order partial derivatives f_x, f_y are again functions of (x,y) we can define their partial derivatives. These will be called the second order partial derivatives, and there can be four possibilities: $f_{xx}, f_{xy}, f_{yx}, f_{yy}$. An example will clarify: if

$$f(x,y) = x^3 y^2$$

then

$$\begin{aligned} f_{xx} &= 6xy^2, \\ f_{yy} &= 2x^3, \\ f_{xy} &= 6x^2y, \\ f_{yx} &= 6x^2y. \end{aligned}$$

Note that in this example $f_{xy} = f_{yx} = 6x^2y$. This is not a coincidence. In general, the mixed second order patial derivatives (or we just say the mixed partial derivatives) are equal if f is "nice" enough. More specifically, we have

Theorem 1.1. *Clairaut's Theorem*

If f_{xy} and f_{yx} are both continuous function on a disk D then $f_{xy} = f_{yx}$ on this disk.

The only thing you should remember about this theorem is the suffficient condition for the mixed derivative to be equal is that they are continuous. A counter example to Clairaut's Theorem is the function

$$\begin{aligned} f(x,y) &= xy\frac{x^2-y^2}{x^2+y^2}, (x,y) \neq (0,0) \\ &= 0, (x,y) = (0,0). \end{aligned}$$

You can verify that $f_{xy}(0,0)$ and $f_{yx}(0,0)$ exist but are not equal.

2 Differentiability and tangent planes

What does it mean for a function $f(x)$ to be differentiable at a point a? It means $f(x)$ is close enough to a linear function (a line) for x sufficient close to a. More specifically we write

$$f(x) \approx f(a) + f'(a)(x-a),$$

for x close to a. The importance in the above intuition is that x can be close to a in either direction (left or right of a). For example, the function $f(x) = |x|$ is close to a linear function namely x for $x > 0$ but to a different linear function namely $-x$ for $x < 0$. Thus it is not differentiable at $x = 0$.

The corresponding linear function for the multivariate case is a plane. Thus we want to say $f(x,y)$ is differentiable at a point (x_0, y_0) if it is close to the tangent plane $ax + by + cz = d$ for (x,y) close to (x_0, y_0).

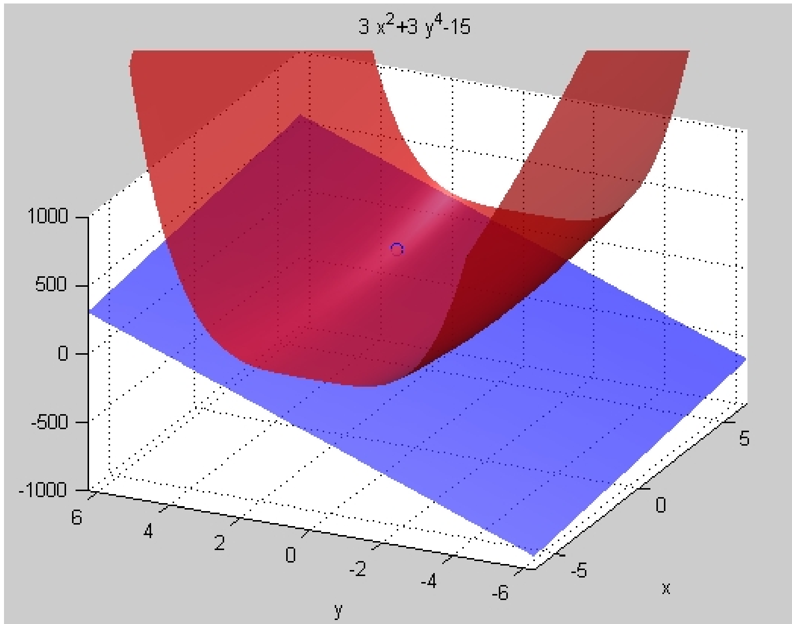


Figure 2.1: A surface and its tangent plane at a point

We need to address two points:

What are a, b, c, d in the coefficients of the tangent plane?

What do we mean by being close to the plane precisely?

To answer the first question, note that if we fix $y = y_0$ then $z = f(x, y_0)$ becomes a curve. This curve can be approximated by the tangent line

$$f(x, y_0) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0).$$

Similarly when we fix $x = x_0$ we have

$$f(x_0, y) \approx f(x_0, y_0) + f_y(x_0, y_0)(y - y_0).$$

So in general, we can hope to have

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{1}$$

The equation

$$z(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

represents a plane so this is our candidate for the tangent plane.

How do we capture closeness from the surface $f(x,y)$ to the tangent plane ? Note that it is not simply enough to require

$$\lim_{(x,y)\to(x_0,y_0)} \Big| f(x,y) - f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \Big| = 0. \qquad (2)$$

This is because ANY other plane that goes through (x_0, y_0) has this property as long as $f(x,y)$ is continuous at (x_0, y_0). Denoting the tangent plane as $p(x,y)$ then the appropriate requirement is

$$\lim_{t\to t_0} \frac{|f(\mathbf{r}(t)) - p(\mathbf{r}(t))|}{t - t_0} = 0, \qquad (3)$$

where $\mathbf{r}(t)$ be an arbitrary differentiable curve such that $\mathbf{r}(t_0) = (x_0, y_0)$.

Example: Let $f(x,y) = x^2 + y^2$. Then the tangent surface to $f(x,y)$ at $(1,1)$ is

$$p(x,y) = 2 + 2(x-1) + 2(y-1).$$

Now let $\mathbf{r}(t) = \big\langle x(t), y(t) \big\rangle$ be an arbitrary differentiable curve. Then

$$\begin{aligned} f(\mathbf{r}(t)) - p(\mathbf{r}(t)) &= x(t)^2 + y(t)^2 - 2 - 2(x(t)-1) - 2(y(t)-1) \\ &= (x(t)^2 - 2x(t) - 1) + (y(t)^2 - 2y(t) - 1). \end{aligned}$$

Since $x(t)$ is differentiable at t_0 and $x(t_0) = 1$ we have

$$\lim_{t\to t_0} \frac{x(t)^2 - 2x(t) - 1}{t - t_0} = \lim_{t\to t_0} 2x(t)x'(t) - 2x'(t) = 0,$$

by L'Hospital rule. Similar conclusion holds for $y(t)$. Thus we have showed that (3) is true for $f(x,y) = x^2 + y^2$ and $p(x,y) = 2 + 2(x-1) + 2(y-1)$.

We conclude by showing for a "wrong" plane (3) does not hold. Indeed consider the plane $p(x,y) = 2 + 3(x-1) + 4(y-1)$. then

$$\begin{aligned} f(\mathbf{r}(t)) - p(\mathbf{r}(t)) &= x(t)^2 + y(t)^2 - 2 - 3(x(t)-1) - 4(y(t)-1) \\ &= (x(t)^2 - 3x(t) + 2) + (y(t)^2 - 4y(t) + 3). \end{aligned}$$

Again applying L'Hospital rule we have

$$\begin{aligned} \lim_{t\to t_0} \frac{x(t)^2 - 3x(t) + 2}{t - t_0} &= \lim_{t\to t_0} 2x(t)x'(t) - 3x'(t) = -1, \\ \lim_{t\to t_0} \frac{y(t)^2 - 4y(t) + 3}{t - t_0} &= \lim_{t\to t_0} 2y(t)y'(t) - 4y'(t) = -2. \end{aligned}$$

Thus (3) is NOT true for the plane $p(x,y) = 2 + 3(x-1) + 4(y-1)$.

We say a function $f(x,y)$ is differentiable at (x_0, y_0) if (2) holds. Note also that we do not have a notion of THE derivative of $f(x,y)$ at (x_0, y_0). You have already seen two candidates of it: $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$. Thus in general we can only talk about derivative of $f(x_0, y_0)$ as we approach (x_0, y_0) from a particular direction. This is the notion of directional derivative, which will be dicussed in the next lecture.

We conclude by stating an abstract result that gives an easier way to decide when $f(x,y)$ is differentiable at (x_0, y_0): f is differentiable at (x_0, y_0) if there is a disk D around (x_0, y_0) so that f_x and f_y exist and are continuous on D.

2.1 Tangent plane and linear approximation

The equation (1) has a pratical application, that is linear approximation of a function at a point. Suppose we want to calculate a complicated expression: $(3.99)^3(1.01)^4(1.98)^{-1}$. This one is not easily done by hand (which is a prolem back in the day when computer is not as readily available). However, we can approximate it by the tangent plane to the function $f(x,y,z) = x^3y^4z^{-1}$ at the point $(x_0, y_0, z_0) = (4, 1, 2)$. The advantage of linear approximation is that it is easy to compute linear expression. Indeed we have

$$f(x,y,z) \approx f(4,1,2) + f_x(4,1,2)(x-4) + f_y(4,1,2)(y-1) + f_z(4,1,2)(z-2),$$

for $(x,y,z) = (3.99, 1.01, 1.98)$ which is sufficiently close to $(4,1,2)$. Thus the problem boils down to computing f_x, f_y, f_z at $(4,1,2)$. Since we have the explicit formula for the partial derivatives, the approximation can be easily done.

We remark that from a computational point of view, the computation of f_x, f_y, f_z may not be trivial (imagine a case where you don't have the explicit formula for f_x, f_y, f_z. Furthermore, this problem is designed for easy approximation as the point $(3.99, 1.01, 1.98)$ is close to a nice point $(4,1,2)$. In reality we may have to evaluate at a point where there is no "nice point" close by, for example at $(\pi, \sqrt{2}, \sqrt{3})$. In this case some other method besides linear approximation is necessary.

References

[1] Carlen, Eric. Multivarible Calculus, Linear Algebra and Differential Equations, Chapter 1-7.