Functions from $\mathbb{R}^n$ to $\mathbb{R}$

Math 251

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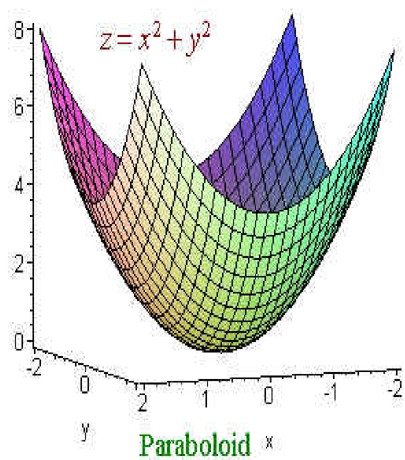
1 Introduction

In this chapter, we study functions that map from $\mathbb{R}^n$ to $\mathbb{R}, n = 2, 3$. The most common interpretation is to view the domain as the $\mathbb{R}^2$ plane or the $\mathbb{R}^3$ space, and the function as giving the property of the corresponding point in the plane or in the space. For example, we can look at $r(x, y) = \sqrt{x^2 + y^2}$ or $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ as the distance from a point to the origin.

When $n = 2$, it is also very commmon to view the mapping $f(x, y)$ as describing the height z of a point (x, y) on the plane. In this case, we have the graph of a surface $\langle x, y, f(x, y)\rangle$ in $\mathbb{R}^3$. For example, $f(x, y) = x^2 + y^2$ describes a paraboloid in $\mathbb{R}^3$.



2 Traces and level curves

When analyzing the graph of $f(x, y)$, it is helpful to see the behavior of the graph as we fixed one of its dimension: $x = a$ or $y = b$ or $f(x, y) = c$. All of these actions have the effect of reducing the dimension of the graph into a curve. Looking at the curves resulting from $f(a, y)$ or $f(x, b)$ is referred to as the *vertical traces* and looking at the curves resulting from $f(x, y) = c$ is referred to as the *horizontal trace* or the *level curve*. For example, you can easily see that the vertical traces of the paraboloid $f(x, y) = x^2 + y^2$ are parabolas while the horizontal traces or level curves are circles.

On the other hand, the level curves of the hyperbolic paraboloid

$$f(x, y) = ax^2 - by^2, a, b > 0$$

are hyperbolas while its vertical traces are parabolas:

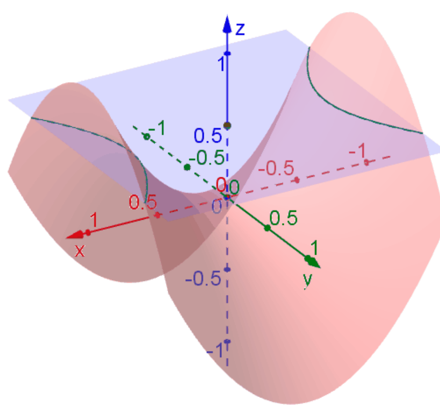


Figure 2.1: Horizontal trace of a hyperbolic paraboloid at $c = 0.5$

For functions from $\mathbb{R}^3$ to $\mathbb{R}$, we can also analyze $f(x, y, z) = c$. This time we refer to it as the level *surfaces*.

3 Limits

3.1 Definition

In Calculus 1, we discussed the concept of

$$y_0 = \lim_{x \to x_0} f(x)$$

for a function $f : \mathbb{R} \to \mathbb{R}$. Intuitively this means as x is close enough to x_0 we have $f(x)$ is closed enough to y_0. Note that here we do not require $f(x_0)$ to be defined (or equal to y_0). This intuition is captured rigorously by the $\epsilon - \delta$ definition: $y_0 = \lim_{x \to x_0} f(x)$ if for all $\epsilon > 0$ we can find $\delta > 0$ so that $|x - x_0| \leq \delta$ implies $|f(x) - f(x_0)| \leq \epsilon$.

What about the case when we have multi-variate functions $f(x, y)$ or $f(x, y, z)$? What does it mean for us to say

$$y_0 = \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})?$$

Notice that we have switched to the vector notation for a generic point $\mathbf{x}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. Clearly we still want the above intuition to hold. We just have to replace the distance between two points in $\mathbb{R} : |x - x_0|$ with the distance between two points in $\mathbb{R}^2$ or $\mathbb{R}^3 : \|\mathbf{x} - \mathbf{x}_0\|$. Thus we also have the following definition for limit: $y_0 = \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ if for all $\epsilon > 0$ we can find $\delta > 0$ so that $|\mathbf{x} - \mathbf{x}_0| \leq \delta$ implies $|f(\mathbf{x}) - f(\mathbf{x}_0)| \leq \epsilon$.

Just as in Calculus 1 where a limit does not have to exist, for example $\lim_{x \to 0} \frac{\cos x}{x}$, here we also have no guarantee whether a particular limit exists before we do some analysis. However, if a limit exists, then it is *unique*. That is we cannot have the case that

$$\begin{aligned} y_0 &= \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \\ y_1 &= \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) \\ y_0 &\neq y_1. \end{aligned}$$

This property is a direct consequence of the definition and will be a very useful criteria to show when *a limit does not exist*, as you shall see.

But before we discuss further about limits, we list out some elementary properties of limits which can be proved using $\epsilon - \delta$ definition. However, such proofs are beyond the scope of this course and can be found in a real analysis course, for example.

Assuming that the limits $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})$ both exist, then

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{x}_0} cf(\mathbf{x}) &= c\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\\ \lim_{\mathbf{x}\to\mathbf{x}_0}[f(\mathbf{x})\circ g(\mathbf{x})] &= \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\circ\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}),\end{aligned}$$

where $\circ$ is a generic operation that can stand for $+, \times, /$. In the case of division, we also need to assume $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) \neq 0$.

3.2 A non-existence example

Example: Does the limit of $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ exist? If so compute it.

Note that the limit definition does not tell us how to do the computation. But first we need to see whether the limit exists or not. One thing you should keep in mind when dealing with limit for multi-variate function is we can get close to a point $\mathbf{x}_0$ along *many* directions. The uniqueness property tells us that if the limit exists then this limit does not depend on which direction we approach $\mathbf{x}_0$ from. We can utilize this property to show the limit does not exist by pointing out two particular directions that give different limit values.

In our case, we can approach $(0,0)$ along the x-axis, the y-axis, or the line $x = y$. These are just 3 cases among infinitely many possibilities. But let's see what we get.

$$\begin{aligned}\lim_{(x,y)\to(0,0),y=0} \frac{x^2}{x^2+y^2} &= \lim_{x\to 0}\frac{x^2}{x^2} = 1\\ \lim_{(x,y)\to(0,0),x=0} \frac{x^2}{x^2+y^2} &= 0\\ \lim_{(x,y)\to(0,0),x=y} \frac{x^2}{x^2+y^2} &= \lim_{x\to 0}\frac{x^2}{2x^2} = 1/2.\end{aligned}$$

So the limit does not exist here by the uniqueness property. Note also that the potential "bad" thing here is the division by 0 at $(x,y) = (0,0)$. This is what causes the ill-behavior of the function $\frac{x^2}{x^2+y^2}$ around this point.

3.3 Computation of limit in general

Example: Does the limit of $\lim_{(x,y)\to(1,1)} x^2 + y^2$ exist? If so compute it.

Here you can have the intuition that nothing "bad" can happen and we really want to say the limit is simply $1^2 + 1^2 = 2$. But at this stage we cannot do so without invoking the $\epsilon - delta$ definition and show that it is true. This is clearly a tedious process. We will revisit this after we introduce the notion of continuous function and justify the direct procedure of "plugging in" to compute limit.

4 Continuity

4.1 Definition

Again, in Calculus 1 we say a function $f(x)$ is continuous at x_0 if the limit as $x \to x_0$ exists and equals to $f(x_0)$. The same intuition and definition applies here. We just replace x_0 and x by its multivariate analog $\mathbf{x}_0$ and $\mathbf{x}$.

Just as in limits, we also have the summation, product and division (at the points where the denominator is non-zero) of continuous functions are continuous. Also composition of continuous function is continuous.

These properties give us a rich "dictionary" of continuous functions. For example, polynomials in single variable are continous. Using the product and summation rules, we conclude polynomials in multi-variable are continuous:

$$f(x, y) = \sum_{i=1}^{N} c_i x^{n_i} y^{m_i}. \tag{1}$$

Quotients of polynomials are continuous where the denominator is not zero:

$$f(x, y) = \frac{g(x, y)}{h(x, y)},$$

where g, h are of the form (1).

Since transcendental functions are continuous (as single variate function), we also have functions of the type $\phi(f(x, y))$ being continuous where ϕ is a transcendental function $\cos, \sin, \exp$ and $f(x, y)$ is of the form (1) by the composition rule.

Example: $f(x, y) = x^2y^3 + 3x^{10}y, g(x, y) = \frac{x^2+y^2-2xy}{x^2+y^2+1}, h(x, y) = e^{x^2+y^2}$ are continuous.

4.2 Evaluation of limits of continuous functions

Since the limit of $f(x, y)$ at $\mathbf{x}_0$ exists and is equal to $f(\mathbf{x}_0)$ if f is continous at $\mathbf{x}_0$, we can use our knowledge of which functions are continuous to evaluate the limit by plugging in. For example, we can now say $\lim_{(x,y)\to(1,1)} x^2 + y^2 = 2$ because

$f(x, y) = x^2 + y^2$ is continuous everywhere. On the other hand, you see why the function $\frac{x^2}{x^2+y^2}$ needs careful analysis for its limit at $(0, 0)$: it is not continuous there.

References

[1] Carlen, Eric. Multivarible Calculus, Linear Algebra and Differential Equations, Chapter 1-7.