Arc length, speed and curvature

Math 251

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1 Arc length and speed

Given $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and a particle travelling along $\mathbf{r}(t)$ during the time interval $a \leq t \leq b$, we want to find the distance travelled by this article. We shall denote the distance travelled during the time interval $[a, b]$ as $s(b) - s(a)$.

If the motion is a straight line motion, that is

$$\mathbf{r}(t) := \mathbf{r}_0 + t\mathbf{v},$$

the answer is clearly $\|\mathbf{v}\|(b - a) = \int_a^b \|\mathbf{r}'(t)\| dt$.

In the general case that $\mathbf{r}(t)$ is a differentiable curve, given a time interval $[t_1, t_2]$ that is sufficiently small, we have discussed in the previous lecture that $\mathbf{r}(t)$ can be approximated by straight line motion:

$$\mathbf{r}(t) \approx \mathbf{r}(t_1) + \mathbf{r}'(t_1)(t - t_1), t_1 \leq t \leq t_2.$$

Thus the distance travelled during the time interval $[t_1, t_2]$ is approximately

$$s(t_2) - s(t_1) \approx \|\mathbf{r}'(t_1)\|(t_2 - t_1).$$

Now given a time interval $[a, b]$, we can partition it into n equal subintervals $a = t_0 < t_1 < \cdots < t_n = b$. If n is lare enough so that each subinterval is sufficiently small, then the total distance travel is approximately

$$s(b) - s(a) \approx \sum_{i=0}^{n-1} \|\mathbf{r}'(t_i)\|(t_{i+1} - t_i).$$

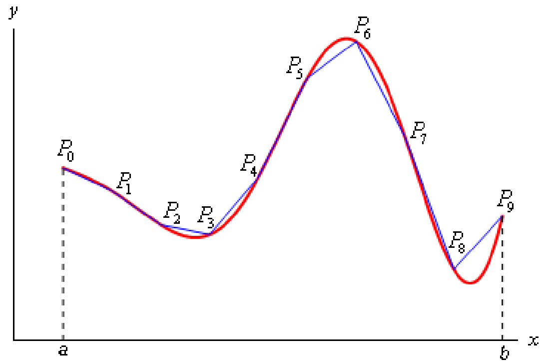

Figure 1.1: Approximation of a curve by line segments

Taking the limit as $n \to \infty$ intuitively this sum will approach the $s(b) - s(a)$. Since the RHS is a Riemann sum, it will approach the Riemann integral of $\|\mathbf{r}'(t)\|$ against dt. That is we have

$$s(b) - s(a) = \int_a^b \|\mathbf{r}'(t)\| dt. \tag{1}$$

A rigorous proof for (1) can be obtained by analyzing the error in the approximation $s(t_{i+1}) - s(t_i) \approx \|\mathbf{r}'(t_{i+1})\|(t_{i+1} - t_i)$ and shows that the sum of the errors over all i goes to 0 as $n \to \infty$. We will just accept it here.

We can consider a motion starting at a reference point t_0 and assign $s(t_0) = 0$. Then we can express s as a function of $t, t \geq t_0$ as followed:

$$s(t) = \int_{t_0}^{t} \|\mathbf{r}'(s)\| ds.$$

We refer to $s(t)$ as the *arc length function* associated with the motion $\mathbf{r}(t)$. Clearly $s'(t) = \|\mathbf{r}'(t)\|$ is the speed of the motion at time t.

1.1 Arc Length Parametrization

We have discussed that there are many ways to parametrize the same curve. In particular, given a curve $\mathbf{r}(t)$ suppose we replace t by a function $t = g(s)$ then effectively we have changed the clock system from t to s. This has the effect of altering the velocity and acceleration of the motion. In particular

$$\frac{d}{ds}\mathbf{r}(g(s)) = \mathbf{r}'(g(s))g'(s).$$

That is we have multiplied a factor $g'(s)$ to the velocity vector $\mathbf{r}'(t)$. If we only care about the *geometry of the curve*, then obviously the speed the particle traversed along the curve is irrelevant. In particular, a choice of parametrization that normalizes this speed to 1 (that is making the motion have unit velocity) has certain nice properties. We call this the *arc length parametrization* and describe it below.

Consider a motion $\mathbf{r}(t)$ starting at t_0 and its associated arc length function $s(t)$. Suppose that $\|\mathbf{r}'(t)\| > 0$ for all $t \geq t_0$ that is the particle never stops, then $s(t)$ is a strictly increasing function. Thus for any given non-negative number s, we can find a unique $t \geq t_0$ so that $s(t) = s$.

This value of t, considers as a function of s, is the inverse function to the arc length function:

$$t(s) = t.$$

It answers the question *How much time have gone by when the distance travelled is s units of length.*

Remark: There is a slight abuse of notation here where we use the letter s as a real number as well as s as the arc length function of t (similarly for t as a real number as t as a function of the arc length s). This avoids the complication of adding more notation of a function $g(s)$ as in the text book. You can always distinguish which meaning we use since a function has the notation $s(t)$ while a number has the notion s.

We recall a important property of the derivative of an inverse function: if g is differentiable and g^{-1} exists then g^{-1} is also differentiable and

$$\frac{d}{ds} g^{-1}(s) = \frac{1}{g'(s)}.$$

The proof is from the chain rule and the identity $g(g^{-1})(s) = s$. It follows from this result that $t(s)$ is also differentiable and $t'(s) = \frac{1}{s'(t)}$ where $s = s(t)$ and $t = t(s)$.

We can then consider the parametrization $\mathbf{r}(t(s))$, which is also referred to as the *arc length parametrization* and investigate its property in the next section.

2 Curvature

2.1 Unit tangent vector

Let $\mathbf{r}(t)$ be a given motion then $\mathbf{r}'(t)$ is its veloctiy or tangent vector at time t. The tangent vector captures how fast and *in which direction* the particle is moving at time t along the curve. Now if we are more interested in a curve's geometric property than how fast the particle is moving along the curve (that is we only care about *the direction* in which the curve is following) then it is better to look at the unit tangent vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}(t)\|}.$$

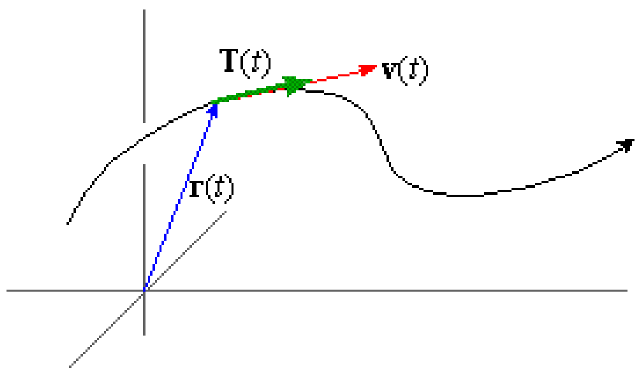

Figure 2.1: Unit tangent vector

If we use arc length length parametrization then automatically the tangent vector is a unit tangent vector:

$$\begin{aligned}
\|\frac{d}{ds}\mathbf{r}(t(s))\| &= \|\mathbf{r}'(t(s))\||t'(s)| \\
&= \|\mathbf{r}'(t(s))\|\frac{1}{|s'(t)|} \\
&= \|\mathbf{r}'(t)\|\frac{1}{\|r'(t)\|} = 1.
\end{aligned}$$

2.2 Curvature

The unit tangent vector $\mathbf{T}(t)$ tells us which direction the curve is following at time t. A straight line motion has $\mathbf{T}(t)$ as a constant unit vector $\mathbf{e_v}$ for all t. Thus for a straight line motion $\|\frac{d}{dt}\mathbf{T}(t)\| = 0$. This is consistent with the fact that a straight line never changes its direction. We say a sraight line has zero curvature.

Now clearly the more quickly a curve changes its direction, the "sharper" the curve. How do we capture this notion precisely? An obvious candidate is $\|\frac{d}{dt}\mathbf{T}(t)\|$ but this quantity is not *intrinsic* to the curve: it depends on an exogenous clock t

that is related to how fast the particle moves. A better "clock" to use is the arc length parametrization $t(s)$. That is we define the curvature of a curve as

$$\kappa(s) = \left\| \frac{d\mathbf{T}(s)}{ds} \right\|,$$

where $\mathbf{T}(s) = \mathbf{T}(t(s)) = \frac{d}{ds}\mathbf{r}(t(s))$ is the unit tangent vector obtained using the arc length parametrization. Intuitively, $\kappa(s)$ measures how fast T is changing with respect to one unit change in arc length, which is exactly what we want.

Example: Arc length of a circle

Let

$$\mathbf{r}(t) = \big\langle R\cos t, R\sin t \big\rangle, 0 \le t \le 2\pi$$

represents the motion a long a circle in $\mathbb{R}^2$ centered at the origin with radius R. The arc length function is $s(t) = Rt$. Thus $t(s) = \frac{s}{R}$. The arc length parametrization is

$$\mathbf{r}(t) = \big\langle R\cos(s/R), R\sin(s/R) \big\rangle, 0 \le t \le 2\pi.$$

Thus

$$\begin{aligned} \mathbf{T}(s) &= \big\langle -\sin(s/R), \cos(s/R) \big\rangle \\ \kappa(s) &= \frac{1}{R}. \end{aligned}$$

This is consistent with the fact that the larger the circle is, the more it looks like a straight line locally and the less its curvature.

2.3 Curvature computation

Arc length parametrization may not be easy to compute explicitly. So *from a comutational point of view* it might be more convenient to work under the orignal clock system t and still obtain the curvature κ *as a function of* t. It can be accomplished by the chain rule:

$$\kappa(s) = \left\| \frac{d}{ds}\mathbf{T}(t(s)) \right\| = \|\mathbf{T}'(t(s))\| \, |t'(s)| = \frac{\|\mathbf{T}'(t(s))\|}{|s'(t)|} = \frac{\|\mathbf{T}'(t(s))\|}{\|\mathbf{r}'(t)\|}.$$

Thus, given a time t we have simply

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}.$$

This formula is already useful as it is. But we can exploit some additional property of $\mathbf{T}(t)$ to write it as

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

This formula may or may not be useful depending on the situation. We present it for completeness. We present its proof in the following section.

2.4 Unit normal vector

Since $\|\mathbf{T}(t)\| = 1$, for all t from our previous discussion we have

$$\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0, \forall t.$$

That is $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$, the direction of motion. We define the unit normal vector $\mathbf{N}(t)$ as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

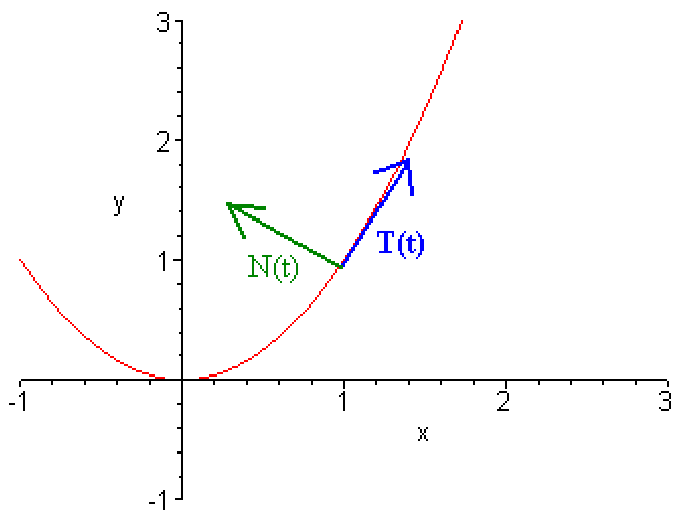

Figure 2.2: Unit normal vector

Again it is a vector that captures some intrinsic geometric property of the curve without paying attention to the speed of the motion. Indeed, denoting $v(t) = \|\mathbf{r}'(t)\|$ the speed of the particle at time t then we have

$$\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t) = \kappa(t)\|\mathbf{r}'(t)\|\mathbf{N}(t) = \kappa(t)v(t)\mathbf{N}(t).$$

We now show

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

For simplicity we will present the computation without the dependent of t, i.e. we write $\mathbf{r}, v, \kappa$ instead of $\mathbf{r}(t), v(t), \kappa(t)$ etc. Observe that

$$\begin{aligned} \mathbf{r}' &= v\mathbf{T} \\ \mathbf{r}'' &= v'\mathbf{T} + v\mathbf{T}' = v'\mathbf{T} + v^2\kappa\mathbf{N}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= v\mathbf{T} \times (v'\mathbf{T} + v^2\kappa\mathbf{N}) \\ &= v^3\kappa\mathbf{T} \times \mathbf{N}. \end{aligned}$$

Since $\mathbf{T}$ is orthogonal to $\mathbf{N}$,

$$\|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{T}\|\|\mathbf{N}\| \sin(\pi/2) = 1.$$

The conclusion follows.

References

[1] Carlen, Eric. Multivarible Calculus, Linear Algebra and Differential Equations, Chapter 1-7.