Vector-valued function calculus

Math 251

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1 Functions from $\mathbb{R}$ to $\mathbb{R}^3$

Now that we have studied vectors $\mathbb{R}^3$, we can study functions whose values are $\mathbb{R}^2$ or $\mathbb{R}^3$ vectors. A simple example of such a function is the position of a particle (for example, an airplane) in 3-d space as a function of time. Because time is one dimensional, precisely these are the functions that map from $\mathbb{R}$ to $\mathbb{R}^2$ or from $\mathbb{R}$ to $\mathbb{R}^3$. This is what the textbook refers to as vector-valued functions.

A popular example of a vector valued function is the spiral:

$$r(t) = \left\langle 4\cos t, 4\sin t, t \right\rangle$$

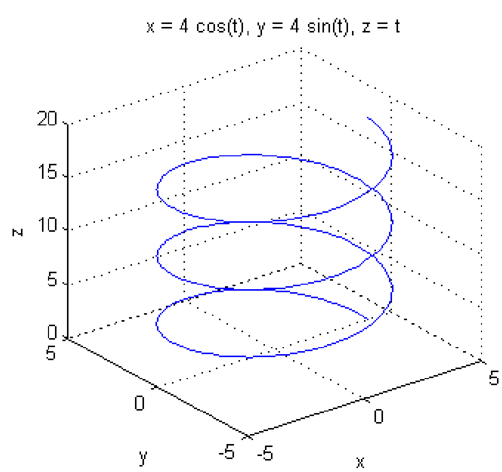

Figure 1.1: Graph of a vector-valued function

Another example of a vector valued function is the parametric representation of a line we covered in the previous chapter:

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t.$$

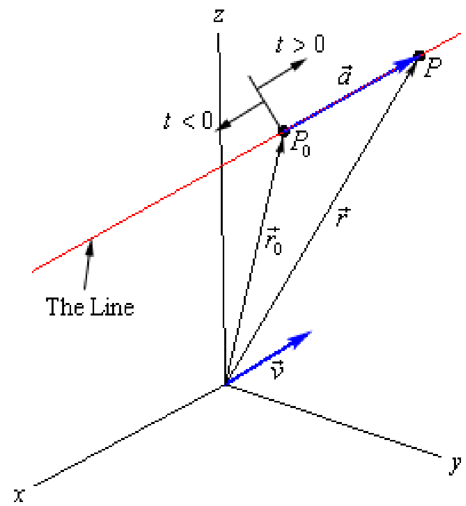

Figure 1.2: A line as a vector-valued function

Remark: It is the convention to refer to functions from from $\mathbb{R}$ to $\mathbb{R}^3$ as vector-valued functions. While we will follow the convention, you should note that this name only refers to its range; but we should also pay attention to the domain of the function. As you can easily imagine, there are functions that map from $\mathbb{R}^3$ to $\mathbb{R}$, which we will study in the next chapter, as well as functions that map from $\mathbb{R}^3$ to $\mathbb{R}^3$, which you may study in another math course.

1.1 Curve as the graph of a vector-valued function

As you probably have noticed, the graph of a vector-valued function is a curve, *and not a surface*. This is because the domain of these functions is *one dimensional.* (Compare vector-valued functions with the parametric representation of a plane, described in Lecture 3, for example).

As t ranges from $-\infty$ to ∞, one can trace *a path* that the particle whose position is represented by $\mathbf{r}(t)$ moves a long the curve. In this way, one can have *the same curve* but *different paths* by using different parametrizations of the same curve. For example,

$$\mathbf{r}_1(t) = \mathbf{r}_0 + \mathbf{v}t$$

and

$$\mathbf{r}_1(t) = \mathbf{r}_0 - \mathbf{v}t$$

for $t \in \mathbb{R}$ describe the same line but one moves along $\mathbf{v}$ and the other moves in the opposite direction of $\mathbf{v}$ as t increases. Another example is $\mathbf{r}_1(t) = \big\langle \cos(t), \sin(t) \big\rangle$ and $\mathbf{r}_2(t) = \big\langle \sin(t), \cos(t) \big\rangle$. You can easily see that they both describe the circle on the plane, centered at the origin with radius 1. As t increases, the first one describes counterclock wise motion while the second one describes clock wise motion.

Example - Parametrization of an ellipse:

Find the parametric representation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Ans: Since $\cos^2(t) + \sin^2(t) = 1$, there must exist $t, 0 \leq t \leq 2\pi$ so that

$$\begin{aligned} x &= a\cos(t) \\ y &= b\sin(t). \end{aligned}$$

Thus $\mathbf{r}(t) = \big\langle a\cos(t), b\sin(t) \big\rangle, 0 \leq t \leq 2\pi$ is the parametric representation of the ellipse.

Example - Parametrization of a tilted ellipse:

Find the parametric representation of the ellipse:

$$5x^2 + 5y^2 - 6xy = 8.$$

Note that this time we cannot simply view this equation as of the form $\cos^2(t) + \sin^2(t) = 1$, for some t as in the previous example. This is because the usual Cartersian coordinate system is not the best to describe this ellipse. Recall in our discussion in the change of coordinate system that the best one to describe this ellipse is the one with the orthonormal basis $\hat{\mathbf{i}} = \frac{1}{\sqrt{2}}\big\langle 1, 1 \big\rangle, \hat{\mathbf{j}} = \frac{1}{\sqrt{2}}\big\langle -1, 1 \big\rangle$.

If $\mathbf{r}(t)$ is the vector representing a point on the ellipse then we have showed that

$$\mathbf{r}(t) = (\mathbf{r}(t) \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{r}(t) \cdot \hat{\mathbf{j}})\hat{\mathbf{j}}. \tag{1}$$

Let

$$\begin{aligned} u &= \mathbf{r}(t) \cdot \hat{\mathbf{i}} \\ v &= \mathbf{r}(t) \cdot \hat{\mathbf{j}}, \end{aligned}$$

we seek an equation to describe u, v. But (1) reads

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}(u - v) \\ y &= \frac{1}{\sqrt{2}}(u + v), \end{aligned}$$

where (x, y) is the coordinate of a point on the ellipse in the original Cartesian system with $\mathbf{i} = \langle 1, 0 \rangle, \mathbf{j} = \langle 0, 1 \rangle$. Plug this into the equation of the ellipse and after simplification, we have

$$u^2 + 4v^2 = 8.$$

Thus following the same argument as the previous example, we can find t so that

$$\begin{aligned} u &= 2\sqrt{2}\cos(t) \\ v &= 2\sin(t). \end{aligned}$$

Thus the paramateric representation of the ellipse is

$$\mathbf{r}(t) = (2\sqrt{2}\cos t)\hat{\mathbf{i}} + (2\sin t)\hat{\mathbf{j}}, \quad 0 \le t \le 2\pi.$$

Example - Intersection of two surfaces: Another way for a curve to arise is as an intersection of two surfaces. For example, it is clear that two planes intersect in a line. We consider here the intersection between a sphere and a plane.

Find the parametric representation of the intersection of the sphere $x^2+y^2+z^2 = 1$ and the plane $x + y + z = 1$.

Ans: By replacing $z = 1 - x - y$ into the first equation, we have

$$2x^2 + 2y^2 - 2x - 2y - 2xy = 0,$$

or

$$x^2 + y^2 - x - y - xy = 0.$$

You can note that this is an equation of an ellipse. This makes sense as we know the curve should be a circle and this ellipse is the projection of this circle on the plane $x + y + z = 1$ onto the xy-plane.

To solve for y, we just re-write the above as a quadratic equation in y:

$$y^2 - (1 + x)y + x^2 - x = 0,$$

which has solution

$$\begin{aligned} y &= \frac{1 + x \pm \sqrt{(1 + x)^2 - 4(x^2 - x)}}{2} \\ &= \frac{1 + x \pm \sqrt{-3x^2 + 6x + 1}}{2}. \end{aligned}$$

Thus we arrive at a parametrization of the curve as

$$\mathbf{r}(t) = \langle t, \frac{1+t \pm \sqrt{-3t^2+6t+1}}{2}, 1-t-\frac{1+t \pm \sqrt{-3t^2+6t+1}}{2} \rangle,$$

where the domain of this function is t such that $-3t^2+6t+1 \geq 0$.

Note that this is a very *awkward* parametrization, which is due to our working with the Cartesian coordinate system $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Since geometrically it is clear that the intersection is a circle on the plane $x+y+z=1$, we should use a coordinate system that is connected with this plane somehow. We consider the solution using a more suitable coordinate system below.

A better solution:

Note that the plane $x+y+z=1$ can be written as

$$\langle 1,1,1 \rangle \cdot \mathbf{r}_1 = 1,$$

where $\mathbf{r}_1$ represents a generic point on the plane. On the other hand the sphere can be written as

$$\|\mathbf{r}_2\| = 1,$$

where $\mathbf{r}_2$ represents a generic point on the sphere.

We use the coordinate system

$$\begin{aligned}
\mathbf{u} &= \frac{1}{\sqrt{3}}\langle 1,1,1 \rangle \\
\mathbf{v} &= \frac{1}{\sqrt{2}}\langle -1,1,0 \rangle \\
\mathbf{w} &= \frac{1}{2}\langle 1,1,-2 \rangle.
\end{aligned}$$

(There is a systematic way to find such as system, called the Gram-Schmidt procedure. We will not discuss it here. The important point is you should recognize $\mathbf{u}$ as the normal vector of the plane, $\mathbf{v}, \mathbf{w}$ are parallel to the plane as they are both orthogonal to $\mathbf{u}$).

Now if $\mathbf{r}(t)$ is the parametric representation of the curve arising from the intersection, then

$$\begin{aligned}
\langle 1,1,1 \rangle \cdot \mathbf{r}(t) &= 1, \\
\|\mathbf{r}(t)\| &= 1.
\end{aligned}$$

Thus $\mathbf{r}(t) \cdot \mathbf{u} = \frac{1}{\sqrt{3}}$ and we have

$$\begin{aligned} \mathbf{r}(t) &= (\mathbf{r}(t) \cdot \mathbf{u})\mathbf{u} + (\mathbf{r}(t) \cdot \mathbf{v})\mathbf{v} + (\mathbf{r}(t) \cdot \mathbf{w})\mathbf{w} \\ &= \frac{1}{\sqrt{3}}\mathbf{u} + (\mathbf{r}(t) \cdot \mathbf{v})\mathbf{v} + (\mathbf{r}(t) \cdot \mathbf{w})\mathbf{w}. \end{aligned}$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are orthornormal, by the Pythagorean theorem we have

$$\|\mathbf{r}(t)\|^2 = \frac{1}{3} + (\mathbf{r}(t) \cdot \mathbf{v})^2 + (\mathbf{r}(t) \cdot \mathbf{w})^2 = 1.$$

Thus we can find t so that

$$\begin{aligned} \sqrt{\frac{2}{3}}\cos(t) &= \mathbf{r}(t) \cdot \mathbf{v} \\ \sqrt{\frac{2}{3}}\sin(t) &= \mathbf{r}(t) \cdot \mathbf{w}. \end{aligned}$$

Thus the parametric representation of the curve is

$$\mathbf{r}(t) = \frac{1}{\sqrt{3}}\mathbf{u} + \sqrt{\frac{2}{3}}\cos(t)\mathbf{v} + \sqrt{\frac{2}{3}}\sin(t)\mathbf{w}, 0 \le t \le 2\pi.$$

2 Calculus of vector-valued functions

2.1 Limit and continuity

Definition 2.1. *A vector-valued function $\mathbf{r}(t)$ approaches the limit $\mathbf{u}$ as t approaches t_0 if*

$$\lim_{t \to t_0} \|\mathbf{r}(t) - \mathbf{u}\| = 0.$$

We denote this as $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{u}$.

Example:

$$\lim_{t \to 0} \langle 4\cos t, 4\sin t, t \rangle = \langle 4, 0, 0 \rangle$$

since

$$\|\langle 4\cos t, 4\sin t, t \rangle - \langle 4, 0, 0 \rangle\|^2 = 16(\cos t - 1)^2 + 16\sin^2(t) + t^2,$$

and clearly

$$\lim_{t \to 0} 16(\cos t - 1)^2 + 16\sin^2(t) + t^2 = 0.$$

You may note that in the above example,

$$\begin{aligned}
\lim_{t\to 0} 4\cos t &= 4 \\
\lim_{t\to 0} 4\sin t &= 0 \\
\lim_{t\to 0} t &= 0.
\end{aligned}$$

Thus we suspect that

$$\lim_{t\to t_0} \big\langle f(t), g(t), h(t)\big\rangle = \big\langle \lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)\big\rangle$$

if the limit makes sense on both sides. That this is true is stated as Theorem 13.1 in the textbook (also see the proof therein). We call this taking the limit *component wise.*

Definition 2.2. *A vector-valued function* $\mathbf{r}(t)$ *is continuous at* t_0 *if*

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

2.2 Differentiability and velocity

Physical motions are usually not only continuous but also differentiable. This is captured by our notion of velocity. To say $\mathbf{r}(t)$ is differentiable roughly means that if you observe $\mathbf{r}(t)$ for a sufficiently short time interval, it looks like motion along a straight line. Thus we first investigate the differentiability of a straight line motion and then of general motion later on.

Straight line motion: Let $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ be a parametrized line. Then for any $s < t$, the distance between $\mathbf{r}(s)$ and $\mathbf{r}(t)$ is

$$\|\mathbf{r}(t) - \mathbf{r}(s)\| = (t-s)\|\mathbf{v}\|.$$

Thus

$$\|\mathbf{v}\| = \frac{\|\mathbf{r}(t) - \mathbf{r}(s)\|}{t-s},$$

and we see that the distance travelled per unit of time is $\|\mathbf{v}\|$. This is what we think of as the *speed* of the motion. The direction of the motion is given by the *constant* unit vector

$$\mathbf{T} = \mathbf{e_v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

Clearly the *velocity* of the straight line motion is given by $\mathbf{v}$. Also note that

$$\lim_{t\to t_0} \frac{\mathbf{r}(t)-\mathbf{r}(t_0)}{t-t_0} = \mathbf{v}.$$

General motion: Now suppose that $\mathbf{r}(t)$ is an arbitrary continuous motion in $\mathbb{R}^3$. Suppose there exists a parametrized line $\mathbf{x}(t) = \mathbf{r}_0 + (t-t_0)\mathbf{v}$, where $\mathbf{r}_0 = \mathbf{r}(t_0)$ so that

$$\lim_{t\to t_0} \frac{\|\mathbf{r}(t)-\mathbf{x}(t)\|}{t-t_0} = 0.$$

We have by triangle inequality

$$\|\frac{\mathbf{r}(t)-\mathbf{r}_0}{t-t_0} - \mathbf{v}\| \le \|\frac{\mathbf{r}(t)-\mathbf{x}(t)}{t-t_0}\| + \|\frac{\mathbf{x}(t)-\mathbf{r}_0}{t-t_0} - \mathbf{v}\|$$

Because

$$\lim_{t\to t_0} \|\frac{\mathbf{x}(t)-\mathbf{r}_0}{t-t_0} - \mathbf{v}\| = 0,$$

we have

$$\lim_{t\to t_0} \frac{\mathbf{r}(t)-\mathbf{r}_0}{t-t_0} = \mathbf{v}.$$

The above limit can be interpreted as

$$\mathbf{r}(t) \approx \mathbf{r}_0 + \mathbf{v}(t-t_0),$$

for t sufficiently close to t_0. In this sense we say the motion of $\mathbf{r}(t)$ is approximately the straight line motion starting at $\mathbf{r}_0$ with velocity $\mathbf{v}$ around t_0. We say that $\mathbf{r}(t)$ is *differentiable* at t_0; the *derivative* of $\mathbf{r}(t)$ at t_0 or the *tangent vector* of $\mathbf{r}(t)$ at t_0 is $\mathbf{v}$ and the *tangent line* to $\mathbf{r}(t)$ at $\mathbf{r}_0$ is $\mathbf{r}_0 + \mathbf{v}(t-t_0)$.

2.3 Computation of derivative

Suppose $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is differentiable at t_0 and its derivative at t_0 is $\mathbf{r}(t_0) = \langle a, b, c\rangle$. From the above discussion, it means that

$$\lim_{t\to t_0} \frac{\langle f(t), g(t), h(t)\rangle - \langle f(t_0), g(t_0), h(t_0)\rangle}{t-t_0} = \langle a, b, c\rangle.$$

But since the limit can be taken componentwise, it means that

$$\lim_{t\to t_0} \frac{f(t)-f(t_0)}{t-t_0} = a.$$

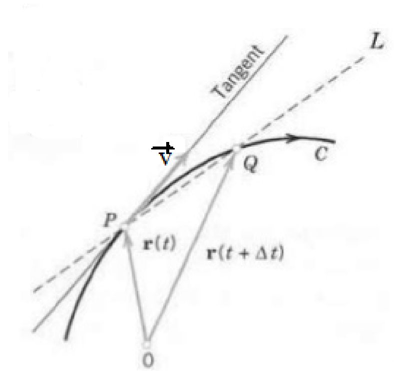

Figure 2.1: Tangent of a vector-valued function

Thus f is differentiable at t_0 and its derivative is a. Similar conclusion holds for g and h. Thus $\mathbf{r}(t)$ is differentiable if and only if all of its components are differentiable (as scalar functions). In this case, we can also take derivative *component wise.* We write

$$\mathbf{r}'(t) = \big\langle f'(t), g'(t), h'(t)\big\rangle$$

2.4 Differentiation rules

Assume that $\mathbf{r}_1(t), \mathbf{r}_2(t)$ are differentiable. Then

$$\begin{aligned}
\frac{d}{dt}\big(\mathbf{r}_1(t) + \mathbf{r}_2(t)\big) &= \mathbf{r}_1'(t) + \mathbf{r}_2'(t) \\
\frac{d}{dt}\big(c\mathbf{r}_1(t)\big) &= c\mathbf{r}_1'(t), \text{ for all scalar } c \\
\frac{d}{dt}\big(f(t)\mathbf{r}_1(t)\big) &= f'(t)\mathbf{r}_1(t) + \mathbf{r}_1'(t)f(t) \text{ for all scalar function} f(t) \\
\frac{d}{dt}\mathbf{r}\big(f(t)\big) &= \mathbf{r}'(f(t))f'(t) \text{ for all scalar function} f(t) \\
\frac{d}{dt}\big(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\big) &= \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t) \\
\frac{d}{dt}\big(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\big) &= \mathbf{r}_1(t)\times\mathbf{r}_2'(t) + \mathbf{r}_1'(t)\times\mathbf{r}_2(t).
\end{aligned}$$

All of these results can be proved by applying the definition and taking the limit component wise. As a particular application, we have the following lemma

Lemma 2.3. *Let $\mathbf{r}(t)$ be a curve such that $\|r(t)\| = c$, a constant for all t. Then $\mathbf{r}(t)$ is always orthogonal to its tangent vector $\mathbf{r}'(t)$.*

Proof. Since $\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = c$ we have

$$0 = \frac{d}{dt}\big(\mathbf{r}(t) \cdot \mathbf{r}(t)\big) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

The conclusion follows.

An interpretation of this lemma is the well-known geometry fact that the tangent to a circle is orthogonal to its radius.

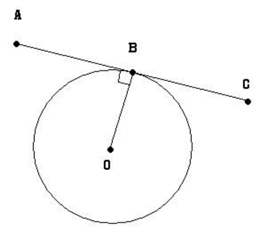

Figure 2.2: Orthogonality between tangent and radius of a circle

Next, we consider the uniform circular motion

$$\mathbf{r}(t) = \rho(\cos(\omega t)\mathbf{u} + \sin(\omega t)\mathbf{v}), t \in \mathbb{R},$$

where $\mathbf{u}, \mathbf{v}$ are two arbitrary orthonormal vectors. This is circular motion with radius ρ, angular speed ω (since the period of motion is $\frac{2\pi}{\omega}$) on a plane determined by $\mathbf{u}, \mathbf{v}$.

We compute that

$$\mathbf{r}'(t) = \rho\omega\Big(-\sin(\omega t)\mathbf{u} + \cos(\omega t)\mathbf{v}\Big).$$

Thus $\|\mathbf{r}'(t)\| = \rho\omega$ is a constant. Applying the above lemma we have

$$\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0.$$

By Newton's second law, $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a} = \mathbf{r}''(t)$. This says that the force acting on a uniform circular motion is along the radius (since it is orthogonal to the tangent). Indeed it points toward the center by explicit computation of $\mathbf{r}''(t)$ and notice that its direction is opposite of $\mathbf{r}(t)$.

References

[1] Carlen, Eric. Multivarible Calculus, Linear Algebra and Differential Equations, Chapter 1-7.