Lines and Planes in $\mathbb{R}^3$

Math 251

September 17, 2015

1 Planes in $\mathbb{R}^3$

1.1 Representation by equations

Just as two points P, Q, or a vector $\overrightarrow{PQ}$ uniquely determine a line, three points P, Q, R, or two vectors $\overrightarrow{PQ}, \overrightarrow{PR}$ uniquely determine a plane in $\mathbb{R}^3$.

In lecture 1, we viewed a line as determined by a point P_0 on the line and a directional vector $\mathbf{v}$. A similar point of view can also be developed for the plane. We can determine a plane $\mathcal{P}$ by a point P_0 that is on the plane and a "directional" vector $\mathbf{n}$ that is orthogonal to $\mathcal{P}$ (that is, orthogonal to every line on $\mathcal{P}$). We refer to $\mathbf{n}$ as the *normal vector* of $\mathcal{P}$.

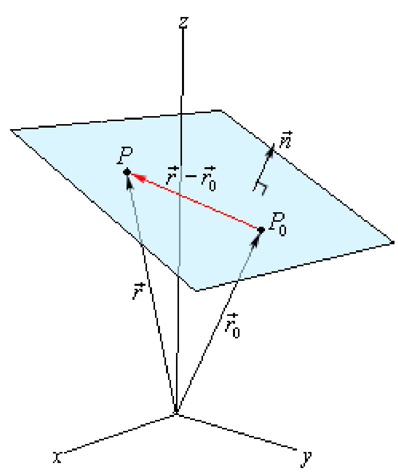

Figure 1.1: A plane and its normal vector

We have the following result:

Proposition 1.1. *Let $\mathcal{P}$ be the plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$. Then any point $P = (x, y, z)$ on $\mathcal{P}$ has to satisfy*

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0. \tag{1}$$

Example 1.2. *Consider the plane passing through the point $P_0 = (3, 2, 1)$ with normal vector $\mathbf{n} = \langle 1, 2, 3\rangle$. Then any point (x, y, z) on this plane must satisfy*

$$(x - 3) + 2(y - 2) + 3(z - 1) = 0,$$

that is

$$x + 2y + 3z = 10.$$

Remark: The above example shows that, the collection of points (x, y, z) satisfying the linear equation

$$ax + by + cz = d$$

is a plane in $\mathbb{R}^3$, with normal vector $\langle a, b, c\rangle$. To completely determine the plane, we just have to find a point P_0 that is on the plane. But that is easy. Since it cannot be the case that $a = b = c = 0$, WLOG suppose $a \neq 0$. Then $P_0 = (d/a, 0, 0)$ belongs to this plane.

1.2 Parametric representation of planes in $\mathbb{R}^3$

In the above subsection we describe a plane via a linear equation $ax + by + cz = d$. This description relies on a point P_0 on the plane and a *normal* vector $\mathbf{n}$ to the plane. On the other hand, a point P_0 on a plane $\mathcal{P}$ and two vectors $\mathbf{u}, \mathbf{v}$ *parallel* to $\mathcal{P}$ also completely describe $\mathcal{P}$. This point of view gives us the parametric representation of a plane (compared with the parametric representation of a line).

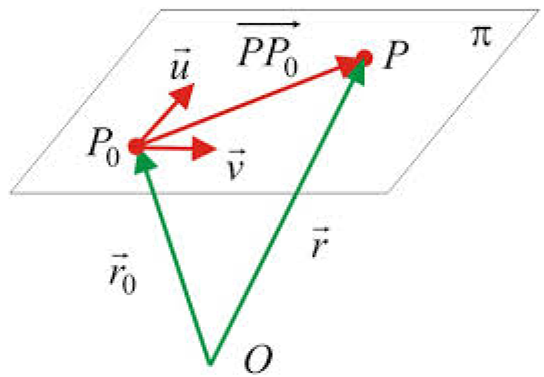

Figure 1.2: A point and two vectors determine a plane

For example, consider the plane $x+2y+z=10$. Then a point $P = \langle x, y, z\rangle$ (here we use the vector representation of P) belongs to the plane if and only if

$$P = \langle x, y, 10-2y-x\rangle \quad = \quad \langle 0,0,10\rangle + x\langle 1,0,-1\rangle + y\langle 0,1,-2\rangle.$$

Here we have $P_0 = (0,0,10)$ belonging to the plane and $\mathbf{u} = \langle 1,0,-1\rangle, \mathbf{v} = \langle 0,1,-2\rangle$ parallel to the plane. To verify that $\mathbf{u}, \mathbf{v}$ are parallel to the plane, we just need to check $\mathbf{u}\cdot\mathbf{n} = 0$, where $\mathbf{n} = \langle 1,2,1\rangle$ is the normal vector to the plane. Similarly for $\mathbf{v}$.

Thus any point P on the plane can be written as $\langle s, t, 10-2s-t\rangle$ where $s, t \in \mathbb{R}$. We refer to this as the parametric representation of a plane.

2 Equations for lines in $\mathbb{R}^3$

Let (x, y, z) be the collection of points belonging to a line $\mathcal{L}$ in $\mathbb{R}^3$. Then we have showed that (x, y, z) can be represented as $(x_0 + tv_1, y_0 + tv_2, z_0 + tv_3), t \in \mathbb{R}$ where $P_0 = (x_0, y_0, z_0)$ is a point on the line and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is a directional vector of the line. This is of course the parametric representation of a line in $\mathbb{R}^3$.

Can we represent lines by equations as we did above with planes? The answer is yes. Indeed, observe that for any point (x, y, z) on the line, the vector $\langle x-x_0, y-y_0, z-z_0\rangle$ must be *parallel* with the directional vector $\mathbf{v}$. Thus we have

$$\mathbf{v} \times \langle x-x_0, y-y_0, z-z_0\rangle = \mathbf{0}.$$

This is the equation of a plane in $\mathbb{R}^3$.

Denoting $\mathbf{x} = \langle x, y, z \rangle$ as a generic point on the line $\mathcal{L}$, we can re-write the above equation as

$$\mathbf{v} \times \mathbf{x} = \mathbf{d}, \tag{2}$$

where $\mathbf{d} = \langle x_0, y_0, z_0 \rangle \times \mathbf{v}$.

You should note the similarity between this and the equation of a plane (1). Note the crucial differences that the equation of a line involves *a cross product* (while that for plane involves a *dot product*). Second, the RHS of the equation of a line is a *vector* (while that of a plane is *a real number*).

Two vectors are equal if and only if their components are equal. Thus (2) is actually a *system of equation*. We demonstrate with an example.

Example 2.1. *Let* $\mathbf{v} = \langle 1, -2, 1 \rangle$ *and* $\mathbf{d} = -\langle 1/2, 1, 5/2 \rangle$, *the system of equations*

$$\mathbf{v} \times \mathbf{x} = \mathbf{d}$$

is

$$\begin{aligned} -y - 2z &= -1/2 \\ x - z &= 1 \\ 2x + y &= 5/2, \end{aligned} \tag{3}$$

since $\langle 1, -2, 1 \rangle \times \langle x, y, z \rangle = \langle -y - 2z, x - z, 2x + y \rangle$.

You can check that this system has the solution

$$(x, y, z) = (1 + t, 1/2 - 2t, t), t \in \mathbb{R},$$

which is the parametric equation of a line.

Remark: Observe that (3) describes the intersection of three planes in $\mathbb{R}^3$. This intersection can be *empty* if the RHS is not chosen carefully (see figure 2.1). Indeed, if $\mathbf{d}$ is not orthogonal to $\mathbf{v}$ then (2) cannot have a solution. On the other hand, the intersection of two planes is sufficient to describe a line, so one (and only one) of the equations of (3) is redundant (Another way to say this is the rank of the system (3) is always 2). This can be showed rigorously but we will skip the proof.

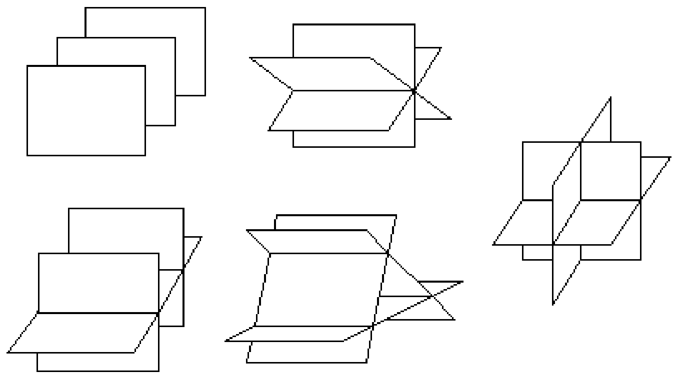

Figure 2.1: Possible intersections of three planes

3 Distance from a point

3.1 Distance from a point to a line

Given a point P and a line $\mathcal{L} : \mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$, we want to know : a) the distance between P and $\mathcal{L}$ and b) the point P' on the line where this distance is achieved.

To understand the problem, you should observe that for any point Q on the line, the distance PQ is defined. However, there is one point P' where this distance is minimized. It also happens that at this point P', PP' is orthogonal to $\mathcal{L}$.

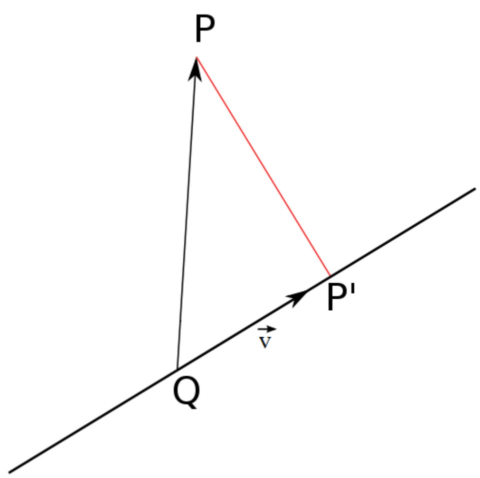

Figure 3.1: Distance from a point to a line

By this observation, to answer a), we note that since $\mathbf{x}_0$ is a point on $\mathcal{L}$, the distance between P and $\mathcal{L}$ is just the orthogonal component of $\mathbf{p} - \mathbf{x}_0$ along the directional vector $\mathbf{v}$: $\|(\mathbf{p} - \mathbf{x}_0)_\perp\|$, where $\mathbf{p}$ is the vector representation of P. Using our previous result on dot product, we thus have

$$\|(\mathbf{p} - \mathbf{x}_0)_\perp\| \;=\; \|(\mathbf{p} - \mathbf{x}_0) - ((\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{e_v})\mathbf{e_v}\|\,,$$

where again $\mathbf{e_v} := \frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is the unit vector in the direction of $\mathbf{v}$.

It would also follow that the point P' on $\mathcal{L}$ that achieves this minimum distance is

$$P' = \mathbf{x}_0 + ((\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{e_v})\mathbf{e_v}.$$

Problem: Show that for any other point Q on $\mathcal{L}$, the distance $PP' \leq PQ$.

3.2 Distance from a point to a plane

Given a point P and a plane $\mathcal{P}$ we want to know : a) the distance between P and $\mathcal{P}$ and b) the point P' on the plane where this distance is achieved.

Recall that a plane is uniquely determined by a point $\mathbf{x}_0$ on the plane and its normal vector $\mathbf{n}$. You should also observe that for any point Q on the plane, the distance PQ is defined. However, there is one point P' where this distance is minimized. It also happens that at this point P', PP' is orthogonal to $\mathcal{P}$.

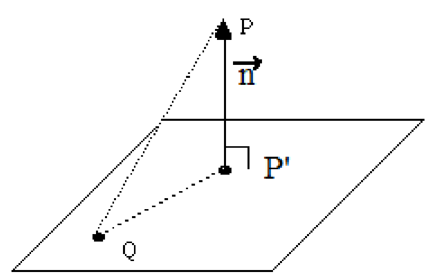

Figure 3.2: Distance from a point to a plane

By this observation, to answer a), we note that since $\mathbf{x}_0$ is a point on $\mathcal{L}$, the distance between P and $\mathcal{L}$ is just the *parallel* component of $\mathbf{p} - \mathbf{x}_0$ along the normal vector $\mathbf{n}$: $\|(\mathbf{p} - \mathbf{x}_0)_{||}\|$, where $\mathbf{p}$ is the vector representation of P.

Using our previous result on dot product, we thus have

$$\|(\mathbf{p} - \mathbf{x}_0)_{||}\| \;=\; |(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{e_n}|\,,$$

where again $\mathbf{e_n} := \frac{1}{\|\mathbf{n}\|}\mathbf{n}$ is the unit vector in the direction of $\mathbf{n}$.

It would also follow that the point P' on $\mathcal{L}$ that achieves this minimum distance is

$$P' = \mathbf{x}_0 + (\mathbf{p} - \mathbf{x}_0)_\perp = \mathbf{p} - ((\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{e_n})\mathbf{e_n},$$

where $(\mathbf{p} - \mathbf{x}_0)_\perp$ is the orthogonal projection of $(\mathbf{p} - \mathbf{x}_0)$ with respect to $\mathbf{n}$.

References

[1] Carlen, Eric. Multivarible Calculus, Linear Algebra and Differential Equations, Chapter 1-7.