

Vector products

Math 251

September 19, 2015

1 Overview

In the previous lecture we have studied two basic operations involving vectors: scalar multiplication and vector addition. It is helpful to remember the input and output of these operations.

Operation	Input	Output
Scalar multiplication	A real number t and a vector \mathbf{v}	A vector $t\mathbf{v}$
Vector addition	Two vectors \mathbf{u}, \mathbf{v}	A vector $\mathbf{u} + \mathbf{v}$

In this lecture we will study two additional operators, whose input are two vectors \mathbf{u}, \mathbf{v} . In this way, they can be viewed as “product” of two vectors. Pay attention: the output of the dot product is *a real number* while the output of the cross product is *a vector*. They are essential tools to understand the three dimensional geometry, i.e. they capture notions as angle, orthogonality (dot product) and area (cross product). You may not appreciate these features since we can draw picture in 3-d and study the geometry in the Euclidean approach. Thus we want to mention two significant “benefits” of these products . First, they allow us to capture *algebraically* our *geometric* intuition. This is essentially the common feature of *analytic geometry*, starting from Descartes. It allows us to solve certain problems that might be more challenging to do using pure geometrical concepts. An example is the proof of the triangle inequality.

Second, note that once we go beyond 3-d (and there is no reason why not, we just have to interpret our vector differently) the spatial intuition that we have will be lost. Yet the mathematical concepts of dot product still makes sense. Thus it generalizes the geometry of 3-d to multi dimensional spaces.

Operation	Output	Geometry association
Dot product	A real number	Cosine, angle between \mathbf{u}, \mathbf{v}
Cross product	A vector $\mathbf{u} \times \mathbf{v}$	Sine, area, volume, a vector orthogonal to \mathbf{u}, \mathbf{v}

2 Dot product

Definition 2.1. Let \mathbf{u}, \mathbf{v} be two vectors in \mathbb{R}^3 . Then the dot product between \mathbf{u} and \mathbf{v} , denoted as $\mathbf{u} \cdot \mathbf{v}$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^3 u_i v_i.$$

Some elementary properties of dot product are

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} \\ (s\mathbf{u} + t\mathbf{v}) \cdot \mathbf{w} &= s\mathbf{u} \cdot \mathbf{w} + t\mathbf{v} \cdot \mathbf{w}. \end{aligned}$$

Note that it does not make sense to talk about $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ since $(\mathbf{u} \cdot \mathbf{v})$ is a scalar and we do NOT have a definition of the dot product between a scalar and a vector. The exception is when all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are scalars or we understand the second \cdot as a scalar multiplication. But in this case using a \cdot in place of a scalar multiplication is a typo that one should avoid.

2.1 The geometry of dot product

2.1.1 Cauchy-Schwartz inequality

You can easily check that $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$. Perhaps more surprisingly is the following Cauchy-Schwartz inequality

Theorem 2.2. For any vectors \mathbf{u}, \mathbf{v}

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

Equality holds if and only if there exists a scalar c such that $\mathbf{u} = c\mathbf{v}$.

As you shall see this inequality plays a *fundamental* role in defining the cosine of angle between two vectors.

Proof. WLOG, we can assume both \mathbf{u}, \mathbf{v} are not the zero vector. Thus it is equivalent to show for \mathbf{u}, \mathbf{v} being two *unit* vectors

$$-1 \leq \mathbf{u} \cdot \mathbf{v} \leq 1,$$

(by replacing \mathbf{u} with $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ and similarly for \mathbf{v}). Consider the identity:

$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v} = 2(1 - \mathbf{u} \cdot \mathbf{v}),$$

where we have used our assumption that $\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v} = 1$ as they are unit vectors. Since $\|\mathbf{u} - \mathbf{v}\|^2 \geq 0$, it must be that $2(1 - \mathbf{u} \cdot \mathbf{v}) \geq 0$ and hence $\mathbf{u} \cdot \mathbf{v} \leq 1$.

Similarly, from $\|\mathbf{u} + \mathbf{v}\|^2 = 2(1 + \mathbf{u} \cdot \mathbf{v})$ we conclude that $-1 \leq \mathbf{u} \cdot \mathbf{v}$.

2.1.2 Angle between two vectors

Let \mathbf{x}, \mathbf{y} be two vectors. Consider the triangle whose vertices are $\mathbf{0}, \mathbf{x}, \mathbf{y}$. Let θ be the angle between \mathbf{x}, \mathbf{y} . Then by the law of cosines:

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta).$$

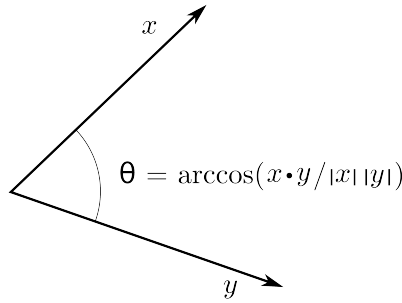


Figure 2.1: Angle between two vectors

On the other hand, we also have

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x} \cdot \mathbf{y}.$$

This shows that

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

Note that this is consistent with the Cauchy-Schwartz inequality, since it implies that

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1,$$

which allows us to associate cosine of an angle with this quantity.

A special case is when $\theta = \frac{\pi}{2}$ and $\cos(\theta) = 0$. Thus we have the following observation: two vectors are *orthogonal* if and only if their dot product is 0.

2.2 Vector components

Given two vectors \mathbf{u} and \mathbf{v} , it is very useful to discuss the *parallel* component of \mathbf{u} along \mathbf{v} and the *orthogonal* component of \mathbf{u} with respect to \mathbf{v} . If you have studied Newtonian mechanics, you can imagine that \mathbf{v} is the direction of movement of an object and \mathbf{u} as the force applied to the object. Then the parallel component directly helps (or opposes) the movement of the object while the orthogonal component does not influence the movement in any direct way (sometimes it get cancelled out by a reaction force from the ground or it may play a role in changing the frictional force if friction is non-zero).

Notation: Given a vector \mathbf{u} , we define $\mathbf{e}_{\mathbf{u}} := \frac{\mathbf{u}}{\|\mathbf{u}\|}$, i.e. the unit vector pointing in the same direction as \mathbf{u} .

Definition 2.3. Let \mathbf{u} and \mathbf{v} be given. We can decompose u into two pieces \mathbf{u}_{\parallel} and \mathbf{u}_{\perp} where

$$\mathbf{u}_{\parallel} = (\mathbf{u} \cdot \mathbf{e}_{\mathbf{v}})\mathbf{e}_{\mathbf{v}}$$

$$\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel}.$$

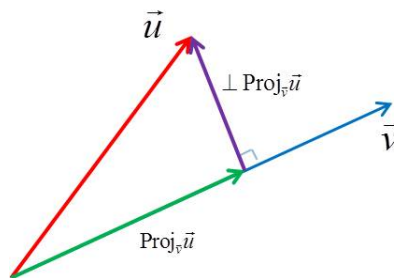


Figure 2.2: Projections of \mathbf{u} w.r.t. \mathbf{v}

Observations:

- $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$ hence the name decomposition.
- \mathbf{u}_{\parallel} is parallel to \mathbf{v} . We say \mathbf{u}_{\parallel} is the *projection* of \mathbf{u} along \mathbf{v} and $\mathbf{u} \cdot \mathbf{e}_{\mathbf{v}}$ is the component of \mathbf{u} along \mathbf{v} .
- $\mathbf{u} \cdot \mathbf{e}_{\mathbf{v}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \cos(\theta)\|\mathbf{u}\|$, where θ is the angle between \mathbf{u}, \mathbf{v} . This justifies the term projection.
- $\mathbf{u}_{\perp} \cdot \mathbf{e}_{\mathbf{v}} = \mathbf{u} \cdot \mathbf{e}_{\mathbf{v}} - (\mathbf{u} \cdot \mathbf{e}_{\mathbf{v}})(\mathbf{e}_{\mathbf{v}} \cdot \mathbf{e}_{\mathbf{v}}) = 0$, thus \mathbf{u}_{\perp} is orthogonal to \mathbf{v} (and hence to \mathbf{u}_{\parallel}).
- Therefore by Pythagorean theorem, $\|\mathbf{u}\|^2 = \|\mathbf{u}_{\parallel}\|^2 + \|\mathbf{u}_{\perp}\|^2$.
- The decomposition is *unique*: suppose there are $\mathbf{u}_1, \mathbf{u}_2$ such that u_1 is parallel to \mathbf{v} , u_2 is orthogonal to \mathbf{v} and $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. Then it also follows that $\mathbf{u}_1 - \mathbf{u}_{\parallel} = \mathbf{u}_2 - \mathbf{u}_{\perp}$. On the other hand, $\mathbf{u}_1 - \mathbf{u}_{\parallel}$ is *orthogonal* to $\mathbf{u}_2 - \mathbf{u}_{\perp}$. The only vector that is orthogonal to itself is the zero vector. Hence $\mathbf{u}_1 = \mathbf{u}_{\parallel}$ and $\mathbf{u}_2 = \mathbf{u}_{\perp}$.

2.3 Orthonormal set and change of coordinate system

Let $\mathbf{v} = \langle x, y, z \rangle$. Observe that with $\mathbf{i} = \langle 1, 0, 0 \rangle, \mathbf{j} = \langle 0, 1, 0 \rangle, \mathbf{k} = \langle 0, 0, 1 \rangle$ we clearly have

$$\begin{aligned}x &= \mathbf{v} \cdot \mathbf{i} \\y &= \mathbf{v} \cdot \mathbf{j} \\z &= \mathbf{v} \cdot \mathbf{k}.\end{aligned}$$

Thus *the coordinate of \mathbf{v} is nothing but the dot product of \mathbf{v} with the corresponding unit vector*. We have the following representation of \mathbf{v} :

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\mathbf{k}.$$

It turns out that there is nothing special about $\mathbf{i}, \mathbf{j}, \mathbf{k}$. They are just some *convenient choice* of coordinate system. There are other possible choices and sometimes it is more desirable to go with a different coordinate system.

For example, the following is the graph of the ellipse satisfying the equation $5x^2 + 5y^2 - 6xy = 8$

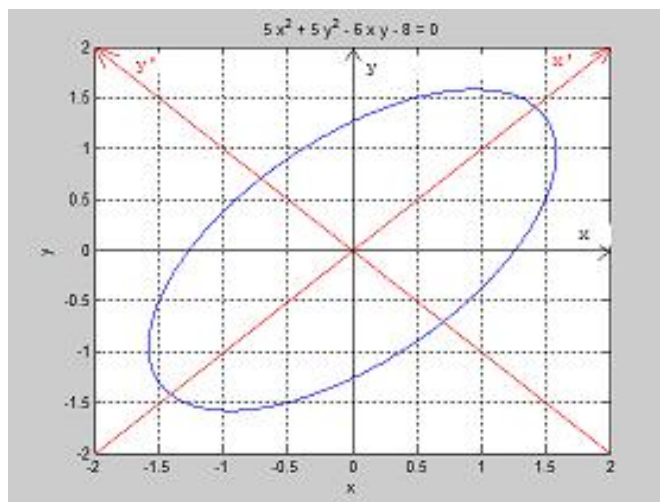


Figure 2.3: A tilted ellipse

The equation $5x^2 + 5y^2 - 6xy = 8$ is not easy to work with. For example, we cannot tell at a glance the length of the major and minor axis of the ellipse. This is because this equation is written using the coordinate system of $\mathbf{i} = \langle 1, 0 \rangle, \mathbf{j} = \langle 0, 1 \rangle$. It turns out if we use the coordinate system $\hat{\mathbf{i}} = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle, \hat{\mathbf{j}} = \langle -\sqrt{2}/2, \sqrt{2}/2 \rangle$ the equation of the ellipse becomes $\frac{x^2}{4} + \frac{y^2}{2} = 1$ which is much easier to deal with. We will explain how to arrive at this equation later when we discuss the topic of parametrized curve.

Now we just want to describe in the abstract different choices of coordinate system and representation of a vector under such a choice. To that end, we define a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ to be an *orthonormal set* if each $\mathbf{v}_i, i = 1, \dots, n$ is a unit vector and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$. In this sense, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal set in \mathbb{R}^3 .

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set in \mathbb{R}^3 . You should visualize that these three vectors define a different coordinate system.

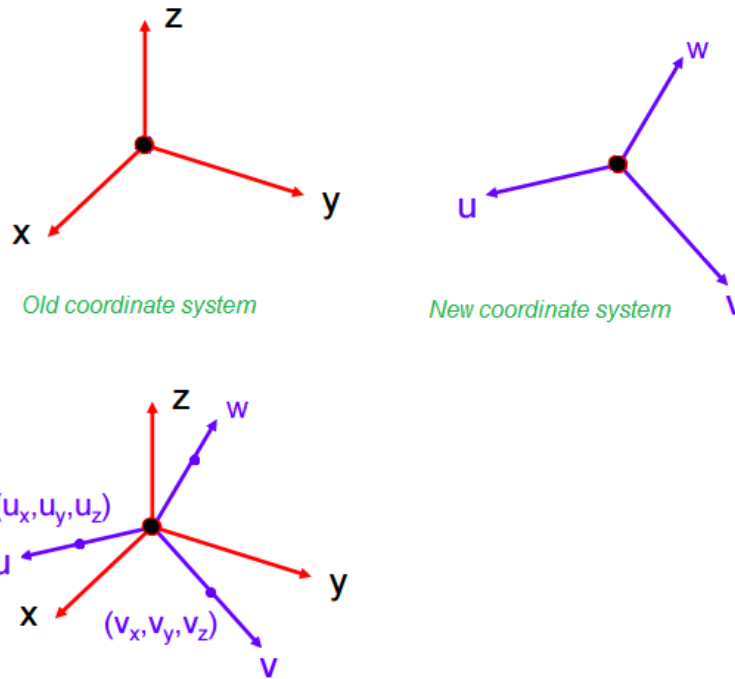


Figure 2.4: Change of coordinate system

We want to represent an arbitrary vector $\mathbf{x} \in \mathbb{R}^3$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. To this end, we try to solve the system of equations:

$$\mathbf{x} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3, \quad (1)$$

where $\alpha_1, \alpha_2, \alpha_3$ are the unknowns to be solved for. You should observe that the above is a 3×3 system, and it may or may not have a unique solution, depending on our choice of $\mathbf{v}_i, i = 1, 2, 3$. In linear algebra, we call any set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that (1) has a unique solution a *linearly independent* set of vectors. We will take for granted here the fact that any orthonormal set of 3 vectors in \mathbb{R}^3 is linearly independent (the proof is offered in a typical linear algebra class).

What is important for us here is the relation $\alpha_i = \mathbf{x} \cdot \mathbf{v}_i, i = 1, 2, 3$. That is

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{v}_1) \mathbf{v}_1 + (\mathbf{x} \cdot \mathbf{v}_2) \mathbf{v}_2 + (\mathbf{x} \cdot \mathbf{v}_3) \mathbf{v}_3. \quad (2)$$

Note that we can only arrive at this equation after we can show that (1) has a unique solution. But if we accept this fact, then the assertion (2) is easy to show,

since

$$\mathbf{x} \cdot \mathbf{v}_1 = \left(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 \right) \cdot \mathbf{v}_1 = \alpha_1,$$

by the orthonormality of the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

This is the representation of \mathbf{x} under the coordinate system $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. We call $\mathbf{x} \cdot \mathbf{v}_i$ the coordinate of \mathbf{x} along the axis $\mathbf{v}_i, i = 1, 2, 3$. This agrees with our geometric intuition since $\mathbf{x} \cdot \mathbf{v}_i$ is just the component of \mathbf{x} along \mathbf{v}_i .

We now show the Pythagorean theorem for \mathbf{x} under the coordinate system $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that

$$0 = \left(\mathbf{x} - \sum_{i=1}^3 (\mathbf{x} \cdot \mathbf{v}_i) \mathbf{v}_i \right) \cdot \left(\mathbf{x} - \sum_{i=1}^3 (\mathbf{x} \cdot \mathbf{v}_i) \mathbf{v}_i \right) = \mathbf{x} \cdot \mathbf{x} - \sum_{i=1}^3 (\mathbf{x} \cdot \mathbf{v}_i)^2,$$

where we again use the orthonormality of the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Thus we have

$$\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = \sum_{i=1}^3 (\mathbf{x} \cdot \mathbf{v}_i)^2$$

3 Cross product

3.1 Cross product and area

Let \mathbf{x}, \mathbf{y} be two vectors. Consider the triangle whose vertices are $\mathbf{0}, \mathbf{x}, \mathbf{y}$. Then one can easily find that the area of this triangle is

$$A = \frac{1}{2} \|\mathbf{x}\| \|\mathbf{y}\| \sin(\theta),$$

where θ is the angle between \mathbf{x} and \mathbf{y} . Thus

$$\begin{aligned} 4A^2 &= \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 (1 - \cos^2(\theta)) \\ &= \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \\ &= (x_2 y_3 - x_3 y_2)^2 + (x_3 y_1 - x_1 y_3)^2 + (x_1 y_2 - x_2 y_1)^2 \\ &= \|\mathbf{x} \times \mathbf{y}\|^2, \end{aligned}$$

where

$$\mathbf{x} \times \mathbf{y} := \langle x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1 \rangle. \quad (3)$$

This motivates us to define

Definition 3.1. Let \mathbf{x}, \mathbf{y} be two \mathbb{R}^3 vectors. The cross product between \mathbf{x} and \mathbf{y} , denoted as $\mathbf{x} \times \mathbf{y}$ is a \mathbb{R}^3 vector defined by equation (3).

Remark:

- An alternative to the RHS of (3) is to write

$$\mathbf{x} \times \mathbf{y} = \det \left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \right),$$

where \det refers to the determinant of a matrix.

- It follows from the above computation that $2A = \|\mathbf{x} \times \mathbf{y}\|$, where A is the area of the triangle whose vertices are $\mathbf{0}, \mathbf{x}, \mathbf{y}$. But then $2A$ is the area of the parallelogram whose vertices are $\mathbf{0}, \mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y}$. Thus we say the parallelogram spanned by \mathbf{x} and \mathbf{y} has area $\|\mathbf{x} \times \mathbf{y}\|$.
- Also from the above computation, we have

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\| \sin(\theta). \quad (4)$$

Thus cross product is related to sine of the angle between two vectors.

- From identity (4) we have the following useful observation: $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u} = c\mathbf{v}$ for some scalar c . Indeed if $\mathbf{u} = c\mathbf{v}$ then by direct computation we can verify $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. On the other hand, if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then (4) tells us that the angle between \mathbf{u} and \mathbf{v} must be 0. That is \mathbf{u} and \mathbf{v} are parallel.
- Some elementary properties that can be verified by basic computations:

- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
- $(t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}) = t(\mathbf{u} \times \mathbf{v})$
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.
- $\mathbf{u} \times \mathbf{v} = (\mathbf{u}_{\parallel} + \mathbf{u}_{\perp}) \times \mathbf{v} = \mathbf{u}_{\perp} \times \mathbf{v}$ since $\mathbf{u}_{\parallel} \times \mathbf{v} = \mathbf{0}$ as we mentioned above.
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$.

The last property sometimes is referred to as the right hand rule, as demonstrated in the following figure

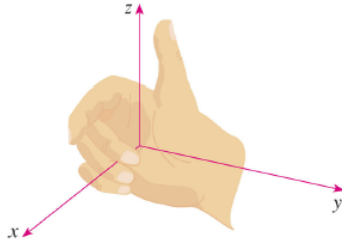


Figure 3.1: The right hand rule

3.2 Cross product and orthogonality

One perhaps fundamental property of cross product is $\mathbf{u} \times \mathbf{v}$ is orthogonal to *both* \mathbf{u} and \mathbf{v} . Before we can show this we need an auxiliary lemma which is interesting by itself.

Lemma 3.2. *We have*

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

Proof. We compute directly that

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3 \\ &= (v_2w_3 - v_3w_2)u_1 + (v_3w_1 - v_1w_3)u_2 + (v_1w_2 - v_2w_1)u_3 \\ &= \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}). \end{aligned}$$

Applying this Lemma we have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{u} = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{u}) = \mathbf{0}.$$

Similarly $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \mathbf{0}$. Thus $\mathbf{u} \times \mathbf{v}$ is orthogonal to both \mathbf{u} and \mathbf{v} .

3.3 Orthogonal component via cross product

Recall the discussion in 2.2. There we defined $\mathbf{u}_\perp = \mathbf{u} - \mathbf{u}_\parallel$. This is a fine definition, but it's a bit non-elegant. It turns out we have a much nicer expression for \mathbf{u}_\perp via cross product. Before we can obtain the formula, we also need to mention another technical lemma, known as Lagrange's identity.

Lemma 3.3. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. We have

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Remark: A way to remember this formula is to note that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is parallel to the plane determined by \mathbf{v} and \mathbf{w} (see more about the discussion on planes in the next lecture). Thus the RHS of Lagrange's identity must be of the form $s\mathbf{v} + t\mathbf{w}$ where s, t are some scalars to be determined.

Proof. By direct computation.

Corollary 3.4. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Then $\mathbf{u}_\perp = -\mathbf{e}_v \times (\mathbf{e}_v \times \mathbf{u})$.

Proof. By the Lagrange's identity we have

$$\begin{aligned} -\mathbf{e}_v \times (\mathbf{e}_v \times \mathbf{u}) &= (\mathbf{e}_v \cdot \mathbf{e}_v)\mathbf{u} - (\mathbf{e}_v \cdot \mathbf{u})\mathbf{e}_v \\ &= \mathbf{u} - (\mathbf{e}_v \cdot \mathbf{u})\mathbf{e}_v = \mathbf{u}_\perp. \end{aligned}$$

Lemma 3.5. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Then $\|\mathbf{u}_\perp\| = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{v}\|}$.

Proof. This can be seen directly (with a picture) from formula (4). Alternatively, we can derive it from

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}_\perp \times \mathbf{v}\| = \|\mathbf{u}_\perp\| \|\mathbf{v}\|,$$

where the last equality also follows from (4) and the fact that $\sin(\pi/2) = 1$.

3.4 Non-associativity of cross product

You may suspect that for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$,

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

However, this is NOT true. The proof of this is also from Lemma (3.3). Indeed, by the Lemma we have

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{v} \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

On the other hand,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = -(\mathbf{v} \times \mathbf{w}) \times \mathbf{u} = -\mathbf{v} \times \mathbf{w} \times \mathbf{u} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{v} \cdot \mathbf{u})\mathbf{w},$$

and indeed they are different.

References

- [1] Carlen, Eric. [Multivariable Calculus, Linear Algebra and Differential Equations, Chapter 1-7.](#)