Surface integrals

Math 251

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1 Parametrized surface

Definition 1.1. *A surface in 3-d is a collection of points* $(x(u,v), y(u,v), z(u,v))$ *where* u, v *are parameters belonging to a certain domain in the* u, v *plane.*

The simplest example of a parametrized surface is given by a function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$. Suppose $(x,y) \in [a,b] \times [c,d]$. Then the collection of points

$$(x, y, f(x,y)), (x,y) \in [a,b] \times [c,d]$$

represents a surface in 3-d and the parameters domain is the rectangle $[a,b] \times [c,d]$.

Example 1.2. *The collection of points* $(x, y, x^2 + y^2), (x,y) \in [-1,1] \times [-1,1]$ *represents the surface of a paraboloid in 3-d with parameters domain* $[-1,1] \times [-1,1]$.

The parameters domain does not have to be rectangular, as the following example makes clear.

Example 1.3. *Consider the upper-half unit sphere with center at the origin. One way to represents it is by the collection of points* $(x, y, \sqrt{1 - x^2 - y^2})$ *but now the domain of* (x,y) *is* $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -1 \leq x \leq 1$, *namely the unit circle on the plane with center at the origin.*

There are obviously more than one way to parametrize a surface. One possible criterion to choose the parametrization is such that the parameter domain is rectangular. For example, a better way to parametrize the upper unit sphere would be via the spherical coordinate.

Example 1.4. *Consider again the upper-half unit sphere with center at the origin. A better parametrization for this surphace is* $(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$ *where* $0 \leq \phi \leq \pi/2, 0 \leq \theta \leq 2\pi$.

2 Tangent plane to a surface at a point

Consider a surface $\mathcal{S} : (x(u,v), y(u,v), z(u,v))$ at a point (u_0, v_0). If we fix u_0 and run along v then we get a curve $(x(u_0,v), y(u_0,v), z(u_0,v))$. The tangent vector of this curve at v_0 is

$$\mathbf{T}_v(u_0,v_0) = \big\langle x_v(u_0,v_0), y_v(u_0,v_0), z_v(u_0,v_0)\big\rangle.$$

Similarly when we fix v_0 and rin along u we obtain another curve and a tangent vector

$$\mathbf{T}_u(u_0,v_0) = \big\langle x_u(u_0,v_0), y_u(u_0,v_0), z_u(u_0,v_0)\big\rangle.$$

The surface $\mathcal{S}$ is said to be normal at (u_0,v_0) if the normal vector

$$\mathbf{n}(u_0,v_0) := \mathbf{T}_u(u_0,v_0) \times \mathbf{T}_v(u_0,v_0)$$

is non-zero.

The tangent plane to the surface $\mathcal{S}$ is the plane through $(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$ with normal vector $\mathbf{n}(u_0,v_0)$.

Example 2.1. *Consider the sphere with radius R and center at the origin with parametrization $(R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi))$ where $0 \le \phi \le \pi, 0 \le \theta \le 2\pi$. We have*

$$\begin{aligned}\mathbf{T}_\theta &= R\big\langle -\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0\big\rangle \\ \mathbf{T}_\phi &= R\big\langle \cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), -\sin(\phi)\big\rangle.\end{aligned}$$

Then the inward normal vector is

$$\begin{aligned}\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_\phi &= -R^2\sin\phi\big\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\big\rangle \\ &= -R^2\sin(\phi)\mathbf{e}_r,\end{aligned}$$

where again $\mathbf{e}_r(x,y,z) := \frac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}}$. We can easily verify that in spherical coordinate, $e_r(R,\theta,\phi) = \big\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\big\rangle$.

Similarly we have the outward normal vector $\mathbf{n} = \mathbf{T}_\phi \times \mathbf{T}_\theta = R^2\sin(\phi)\mathbf{e}_r$.

Remark 2.2. *Note that in the above calculation, the normal vector at the point $(0,0,1)$ (or in spherical coordinate $\phi = 0$) is $\mathbf{0}$. Of course there is a normal vector to the unit sphere at $(0,0,1)$. However, we could not capture it with our calculation* *using spherical coordinate. This is a reflection of a deep fact about the sphere not being equivalent to a flat surface (in this case the flat rectange $0 \le \phi \le \pi, 0 \le \theta \le 2\pi$).*

To see that this is a fundamental geometric fact and not just a flaw with the spherical coordinate, note that in the rectangular coordinate $(x, y, \sqrt{1 - x^2 - y^2}$ we cannot define the normal vector at the boundary $z = 0$.

3 Scalar surface integral

Let $\mathcal{S}(u, v)$ be a surface. Fix a point (u_0, v_0) on $\mathcal{S}$. From (u_0, v_0) we can move a small (signed) distance du in the u direction along the surface $\mathcal{S}$. Approximately we will be at the point

$$\begin{aligned}\mathcal{S}(u_0, v_0) + \mathbf{T}_u(u_0, v_0)du &= \big\langle x(u_0, v_0) + x_u(u_0, v_0)du, y(u_0, v_0) + y_u(u_0, v_0)du, \\ &\quad z(u_0, v_0) + z_u(u_0, v_0)du\big\rangle.\end{aligned}$$

Similarly, if we can a small (signed) distance dv in the v direction along the surface $\mathcal{S}$ and we approximately will be at

$$\begin{aligned}\mathcal{S}(u_0, v_0) + \mathbf{T}_v(u_0, v_0)du &= \big\langle x(u_0, v_0) + x_v(u_0, v_0)dv, y(u_0, v_0) + y_v(u_0, v_0)dv, \\ &\quad z(u_0, v_0) + z_v(u_0, v_0)dv\big\rangle.\end{aligned}$$

In conclusion, if we move along the rectangle

$$(u_0, v_0); (u_0 + du, v_0); (u_0, v_0 + dv); (u_0 + du, v_0 + dv)$$

then along the surface $\mathcal{S}$ we approximately move along the parallelogram

$$\begin{aligned}\mathcal{S}(u_0, v_0) \quad &; \quad \mathcal{S}(u_0, v_0) + \mathbf{T}_u(u_0, v_0)du \\ \mathcal{S}(u_0, v_0) + \mathbf{T}_v(u_0, v_0)dv \quad &; \quad \mathcal{S}(u_0, v_0) + \mathbf{T}_u(u_0, v_0)du + \mathbf{T}_v(u_0, v_0)dv.\end{aligned}$$

This approximation is exact if $\mathcal{S}$ is a plane in 3d, for example. The area of this parallelogram is

$$\begin{aligned}\|\mathbf{T}_u(u_0, v_0)du \times \mathbf{T}_v(u_0, v_0)dv\| &= \|\mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0)\|dudv \\ &= \|\mathbf{n}(u_0, v_0)\|dudv.\end{aligned}$$

Now suppose $\mathcal{S}(u, v)$ is defined for (u, v) in some region $\mathcal{R}$. As we go over all (u, v) in a partition of $\mathcal{R}$ and sum up all the areas of the parallelograms at (u, v) then we obtain approximately the surface area of $\mathcal{S}$. The approximation gets better as the partition becomes finer. Thus we conclude S, the surface area of $\mathcal{S}$ is equal to

$$S = \iint_{\mathcal{R}} \|\mathbf{n}(u,v)\| dudv.$$

For a scalar function $f(u,v)$ defined on $\mathcal{R}$ we can also define

$$\iint_{\mathcal{R}} f(u,v)\|\mathbf{n}(u,v)\| dudv. \tag{1}$$

If $f(u,v)$ is interpreted as the density (charge, mass) of the surface then the double integral (1) captures the total charge or mass of the surface.

4 Vector surface integral

Similar to vector line integral, we also have vector surface integral. Let $\mathcal{F}$ be a vector field and $\mathcal{S}(u,v), (u,v) \in \mathcal{R}$ be a surface in $\mathbb{R}^3$. The unit vector of $\mathcal{S}$ at a point (u_0, v_0) is defined by

$$\mathbf{e_n}(u_0, v_0) = \frac{\mathbf{n}(u_0, v_0)}{\|\mathbf{n}(u_0, v_0)\|}.$$

We note that $\mathbf{e_n}$ can be an outward or inward unit normal vector, depending on how we parametrize the surface. In either way, fixing an orientation of $\mathbf{e_n}$ we can define

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{S} &:= \iint_{\mathcal{R}} \mathbf{F}(\mathcal{S}(u,v)) \cdot \mathbf{e_n}(u,v)\|\mathbf{n}(u,v)\| dudv \\ &= \iint_{\mathcal{R}} \mathbf{F}(\mathcal{S}(u,v)) \cdot \mathbf{n}(u,v) dudv. \end{aligned}$$

If $\mathbf{F}$ is interpreted as a flow, then $\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{S}$ is interpreted as the total (inward or outward) flux through the surface $\mathcal{S}$.