Conservative vector fields and fundamental theorem of line integral

Math 251

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1 Conservative vector field

Definition 1.1. *Let* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a vector field. We say* $\mathbf{F}$ *is a conservative vector field if there exists a function* $V : \mathbb{R}^3 \to \mathbb{R}$ *so that* $\mathbf{F} = \nabla V$*. That is*

$$\mathbf{F}(x, y, z) = \langle V_x, V_y, V_z \rangle.$$

We say $V(x, y, z)$ *is a potential function of* F*.*

Remark 1.2. *It is easy to see that potential function is only unique up to a constant. Indeed, if* $V : \mathbb{R}^3 \to \mathbb{R}$ *is given then* $\nabla V = \nabla(V + 1)$ *is a vector field whose candidate potential functions can be any choice of* $V + c$*, where* c *is a constant.*

Lemma 1.3. *(Necessary criterion for conservative vector field)*
Let $\mathbf{F} = \left\langle F_1, F_2, F_3 \right\rangle$ *be a conservative vector field. Then*

$$\begin{aligned}
(F_1)_y &= (F_2)_x \\
(F_1)_z &= (F_3)_x \\
(F_2)_z &= (F_3)_y.
\end{aligned}$$

Example 1.4. *Denote* $\mathbf{e}_r(x, y, z) = \frac{1}{r}\left\langle x, y, z \right\rangle$*, where* $r = \sqrt{x^2 + y^2 + z^2}$*.* $\mathbf{e}_r(x, y, z)$ *is a unit vector in the direction of* $\left\langle x, y, z \right\rangle$*. We can also think of* $\mathbf{e}_r(x, y, z)$ *as a vector field that assigns the direction of* $\left\langle x, y, z \right\rangle$ *to the point* (x, y, z)*. Let* $\mathbf{F} = \mathbf{e}_r$*. Then* $\mathbf{F}$ *is a conservative vector field.*

Proof: It is easy to check that $V = \sqrt{x^2 + y^2 + z^2}$ *is a potential function for* $\mathbf{F}$*.*

Example 1.5. *Let* $\mathbf{e}_r$ *be the same as the previous example. Let* $\mathbf{F} = -\frac{GmM}{r^2}\mathbf{e}_r$. *Then* $\mathbf{F}$ *is a conservative vector field.*

Proof: It is easy to check that $V = \frac{-GmM}{\sqrt{x^2+y^2+z^2}}$ *is a potential function for* $\mathbf{F}$.

Interpretation: This result shows that the gravitational force is conservative. $\mathbf{F}$ *is the attraction force of a mass* m *at the origin acting on a mass* M *at a point* (x, y, z).

The converse of Lemma (1.3) is true, provided that the vector field $\mathbf{F}$ is defined on a simply connected domain D. A domain is simply connected if for any closed path C in the domain, we can shrink C to a point while remaining in D. In the two dimensional case, a domain with a hole is not simply connected. For example, the unit disk minus the origin is not simply connected.

Example 1.6. *Let* $\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ *be a vector field on the domain* D *that is the unit disk minus the origin. Then* $\mathbf{F}(x, y)$ *is not conservative in* D. *We can check that* $V = \tan^{-1}(\frac{x}{y})$ *satisfies* $\nabla V(x, y) = \mathbf{F}(x, y)$ *if* (x, y) *is not the origin. However, if we let* C *be the unit circle centered at the origin then* $\int_C \mathbf{F} \cdot d\mathbf{s} = 2\pi$, *not 0. The reason is that* $V(x, y)$ *is not defined at* $(0, 0)$. *In polar coordinate,* $V = \theta$, *the angle of the point* (x, y) *with respect to the positive* x *axis. We cannot extend the definition of the angle to the origin* $(0, 0)$ *in such a way that makes* V *continuous there. Indeed,* V *cannot be defined in a continuous way over the whole plane. As we go around one circle on the plane, the angle will increase by* 2π *while we return to the same point.*

2 Fundamental theorem of line integral

Theorem 2.1. *Let* C *be a path from point* P *to point* Q. *Let* $\mathbf{F}$ *be a conservative vector field. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{s} = V(Q) - V(P).$$

Proof. Let $\mathbf{r}(t)$ be a parametrization of C such that $\mathbf{r}(0) = P$ and $\mathbf{r}(T) = Q$. Since F is conservative:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^T \nabla V(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = V(\mathbf{r}(T)) - V(\mathbf{r}(0)) = V(Q) - V(P).$$

Corollary 2.2. *Let* C *be a closed curve, that is* $P = Q$. *If* $\mathbf{F}$ *is a conservative vector field then* $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$. *That is a conservative force does zero work on a closed path.*

Corollary 2.3. *The path integral of a conservative vector field is independent of the actual path taken. That is let* $\mathbf{F}$ *be a conservative vector field and* C_1, C_2 *be two paths from* P *to* Q*. Then* $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s} = V(Q) - V(P)$*.*

Theorem 2.4. *Let* $\mathbf{F}$ *be a vector field that has independent path integral. That is for any* C_1, C_2 *two paths from* P *to* Q $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$*. Then* $\mathbf{F}$ *is a conservative vector field.*

Proof. We prove the case for two dimension $\mathbf{F} = \langle F_1, F_2 \rangle$. Consider the path C_1 from (x, y) to $x + h, y$. By the independence of path property, $\int_C \mathbf{F} \cdot d\mathbf{s} = V(x + h, y)$ for some function V. On the other hand, $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^h F_1(x, y)dx$. Thus

$$V(x + h, y) = \int_0^h F_1(x + u, y)du.$$

This implies that V is differentiable in the x component and $V_x(x+h, y) = F_1(x+h, y)$. That is $V_x(x, y) = F_1(x, h)$ for any x, y. The proof for the y-component is similar.