Path integral

Math 251

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1 Integration of scalar valued function along a path

Let $f(x, y, z)$ be given. We have discussed the triple integral of f over a region $\mathcal{R}$ in $\mathbb{R}^3$:

$$\iint_{\mathcal{R}} f(x, y, z) dx dy dz.$$

Looking at it broadly, the integral is just adding the values of f over a collection of points (x, y, z) (here the region $\mathcal{R}$) multiplying with an appropriate scale factor (here $dxdydz$ - the area differential) so that the sum converges. If this is our point of view, then the triple integral is not the only way to "add" the values of f over a collection of points. We can also sum over the values of f over other choices of regions. If we choose our region as a path C then we will obtain the so-called path integral of f over C. The appropriate scale factor for the path integral is the differential of the arc length ds.

Thus, let $\mathbf{r}(t), 0 \leq t \leq T$ be a given path in $\mathbb{R}^3$. We will also denote this path as C. Recall that the arc length function of $\mathbf{r}(t)$ is

$$s(t) = \int_0^t \|\mathbf{r}'(s)\| ds.$$

This implies that the differential $ds(t)$ is

$$ds(t) = \|\mathbf{r}'(t)\| dt.$$

We define the line integral of $f(x, y, z)$ over the path $\mathbf{r}(t), 0 \leq t \leq T$ as

$$\int_C f(\mathbf{r}) ds = \int_0^T f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt.$$

Note: The ds term on the LHS of the above equality is an arc-length differential, NOT a time differential. Indeed it is a function of time as expressed by the relation $ds(t) = \|\mathbf{r}(t)\|dt$.

If we imagine our path C as made up of material of non-uniform density and f represents the density of the the material at the position (x, y, z) then $\int_C f(\mathbf{r}(t))ds(t)$ is the total mass of this physical "path".

Another point to note is that the line integral is independent of parametrization. That is we can parametrize the path $\mathbf{r}(t)$ in many different ways but the path integrals over the *same* path $\mathbf{r}(t)$ using different parametrization would give the same result. To see why this is true, suppose $\mathbf{r}(t(u)), 0 \leq u \leq U$ is a different parametrization of the path $\mathbf{r}(t), 0 \leq t \leq T$. Here the notation says that the new parametrization is obtained by considering t as a function of u and we assume $t(0) = 0, t(U) = T$. We also suppose $\frac{dt}{du}$ exists. The path integral over the u-parametrization is

$$\begin{aligned}
\int_0^U f(\mathbf{r}(t(u)))\|\frac{d}{du}\mathbf{r}(t(u))\|du &= \int_0^U f(\mathbf{r}(t(u)))\|\mathbf{r}'(t(u))\| \left|\frac{dt}{du}\right| du \\
&= \int_0^T f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt
\end{aligned}$$

by the change of variable formula.

2 Integration of vectored value function over a path

Now consider $\mathbf{F}(x, y, z)$, a map from $\mathbb{R}^3$ to $\mathbb{R}^3$. We interpret $\mathbf{F}(x, y, z)$ as assigning a $\mathbb{R}^3$ vector $\mathbf{F}$ to any point (x, y, z) in $\mathbb{R}^3$. The collection of $\mathcal{F}(x, y, z)$ for $(x, y, z) \in \mathbb{R}^3$ is referred to as a vector field.

An example of $\mathbf{F}$ would be the flow of a river, a magnetic field, a force field etc. Here we restrict the domain of $\mathbf{F}$ over a path $\mathbf{r}(t), 0 \leq t \leq T$ in $\mathbb{R}^3$. We want to add $\mathbf{F}$ over this path. What are meaningful ways to do so?

If we imagine $\mathbf{F}$ as a flow, then we can look at the total flow of $\mathbf{F}$ *along* the path $\mathbf{r}(t)$ (that is in the direction tangential to $\mathbf{r}(t)$ or *directly through* $\mathbf{r}(t)$ (that is in the direction orthogonal to $\mathbf{r}(t)$. An example of the first case is calculating the work done by the force field $\mathbf{F}$ along the path $\mathbf{r}(t)$. An example of the second case is calculating the total flux flown through $\mathbf{r}(t)$ of a velocity field (of some substance).

In particular, if we let $\mathbf{T}(t)$ be the unit tangent vector and $\mathbf{e_n}(t)$ be the unit normal vector of $\mathbf{r}(t)$ (recall that $\mathbf{e}_n(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$) then we can define

$$\begin{aligned}\int_0^T \mathbf{F}(\mathbf{r}) \cdot d\mathbf{s} &= \int_0^T \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t) ds(t) \\ &= \int_0^T \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\| dt \\ &= \int_0^T \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt. \end{aligned} \tag{1}$$

and

$$\int_C (\mathbf{F}(\mathbf{r}) \cdot \mathbf{e_n}) ds = \int_0^T \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} ds(t). \tag{2}$$

Remark: In (1) the notation $d\mathbf{s}$ does NOT refer to an arc length differential. Indeed $d\mathbf{s}(t)$ is a vector differential that is equal to $\mathbf{T}(t)ds(t)$. That is it is in the direction of $\mathbf{T}(t)$ with "length" $ds(t)$.

Also note that in (2) the vector differential is also $\mathbf{n}(t)ds(t)$. That is it is in the direction of $\mathbf{n}(t)$ with "length" $ds(t)$. This has to do with the concept that in physics work equals force times distance (when the force is along the direction of the movement) and the flux equals velocity times length (when the velocity is orthogonal to the length). One can still mathematically define the path integral

$$\int_0^T \mathbf{F}(\mathbf{r}) \cdot d\mathbf{T} = \int_0^T \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t) dt;$$

it will only take on a different physical intepretation (not work or flux).

Lastly, we remark that the sign of the vector path integral is subject to orientation of the path (the direction in which we move along the path) but its absolute value is independent of parametrization. The reason is the direction of the vector field is always the same, but if we travel the path in an opposite direction then the unit tangent vector has an opposite sign with the other unit tangent vector in the other direction.