Change of variables

Math 251

December 4, 2015

1 Change of variable in 1-d

1.1 A linear change of variable example

Consider the integral $\int_2^4 e^{-2x} dx$. We would like to make the change of variable $u = 2x$. Then as x goes from 2 to $4, u$ goes from 4 to 8. Can we conclude

$$\int_2^4 e^{-2x} dx = \int_4^8 e^{-u} du?$$

Not quite. You can verify that the two integrals are not equal. Furthermore, consider the graph of the function $u = 2x$. Suppose we partition the interval $[2, 4]$ into 10 sub-intervals of length 0.2 each. Then there are also 10 corresponding sub-intervals of $[4, 8]$ on the u-axis. However, each u sub-interval has length 0.4, not 0.2. The partial sum over the sub-intervals of $[2, 4]$ is

$$\sum_{i=1}^{10} e^{-2x_i}(x_{i+1} - x_i),$$

where $x_i = 2, 2.2, 2.4, \cdots, 3.8, 4$. The partial sum over the sub-intervals of $[4, 8]$ is

$$\sum_{i=1}^{10} e^{-u_i}(u_{i+1} - u_i),$$

where $u_i = 4, 4.4, 4.8, c \ldots, 7.6, 8$. Thus you see that we have

$$\sum_{i=1}^{10} e^{-2x_i}(x_{i+1} - x_i) = \sum_{i=1}^{10} \frac{1}{2} e^{-u_i}(u_{i+1} - u_i).$$

As we let the partition size on x goes to 0, the partition size on u also goes to 0 and we get

$$\int_2^4 e^{-2x}dx = \frac{1}{2}\int_4^8 e^{-u}du.$$

This was captured in Calculus 1 by the rule $du = 2dx$.

1.2 A nonlinear change of variable example

Now consider the integral $\int_2^4 2xe^{-x^2}dx$. Suppose for the moment that we do not know the anti-derivative of $2xe^{-x^2}$, what can we do?

We can make the change of variable $u = x^2$. Then the limits of the integral become u goes from 4 to 16. Because x is positive on the interval $[2, 4]$, $x = \sqrt{u}$. So we have $2xe^{-x^2} = 2\sqrt{u}e^{-u}$. But what about dx? How can we relate it to du?

Intuitively we may expect (incorrectly) our final result to be $\int_4^{16} 2\sqrt{u}e^{-u}du$. Why is this incorrect? It is because we equate du to dx in this expression. But a little thought shows that it cannot be the case.

Going back to the substitution, by writing $u = x^2$ we mean u is a function of $x : u(x) = x^2$. By dx we mean a "small" change in x. And so because u is a function of x, du, which is a "small" change in u, has to be dependent on dx in a nontrivial way. In particular it cannot be dx.

A graph can simply illustrate this point. Consider the graph of $y = x^2$ on the interval $[2, 4]$. Suppose we devide the interval $[2, 4]$ into 10 subintervals. Then each sub-interval of length 0.2 corresponds to a small change in x. There are also correspondingly 10 subintervals in $[4, 16]$ on the y axis. However, if we look at the corresponding change in y on each subinterval we see that the change is NOT of uniform length. Indeed the change on y, say on $[2.2, 2.4]$ is $(2.4)^2 - (2.2)^2$ and on $[3.8, 4]$ is $4^2 - (3.8)^2$. You can verify that $4^2 - (3.8)^2 > (2.4)^2 - (2.2)^2$.

Indeed, the length of each corresponding subinterval of y can be related to the length of each subinterval of x in the following formula:

$$y_{i+1} - y_i \approx 2x_i(x_{i+1} - x_i),$$

or

$$\Delta y \approx 2x\Delta x.$$

Thus we see that the correct substitution for our integral is

$$\int_4^{16} e^{-u} du.$$

<u>Remark:</u> The analysis above shows that we should write the change of variable as (assuming u is increasing)

$$\int_a^b f(u(x))du(x).$$

This captures the fact that not only the function argument of f depends on x but also the differential u also depends on x. Only in the limiting behavior that $\int_a^b f(u(x))du(x)$ is equal to $\int_{u(a)}^{u(b)} f(u)du$, where the latter is understood as taken independently on the $u-$domain without reference to x.

1.3 Invertibility of the substition map

Consider the integral $\int_{-1}^1 2xe^{-x^2}dx$. It is natural for us to make the substitution $u = x^2$ again. We see that u goes from 0 to 1 as x goes from -1 to 1. Thus the same analysis as above might lead us to conclude

$$\int_{-1}^1 2xe^{-x^2}dx = \int_0^1 e^{-u}du.$$

However, this is NOT correct since the LHS is 0 because the function is odd and the RHS is positive. So where did we go wrong? The answer is when we should be careful making the substition $u = x^2$ over the interval $[-1, 1]$, because $u(x) = x^2$ is NOT invertible over $[-1, 1]$. Indeed, if we split the original integral into

$$\int_{-1}^0 2xe^{-x^2}dx + \int_0^1 2xe^{-x^2}dx$$

and make substitution separately on each interval, we see that it should correspond to

$$-\int_0^1 e^{-u}du + \int_0^1 e^{-u}du = 0.$$

Thus we require that the substitution map *be invertible*. If it is not in the original region like this example, then we break it down to several sub-regions where it is invertible.

We comment on the change of variable

$$\int_{-1}^{0} 2xe^{-x^2}dx = -\int_0^1 e^{-u}du.$$

It is common to explain it as

$$\int_{-1}^{0} 2xe^{-x^2}dx = \int_1^0 e^{-u}du = -\int_0^1 e^{-u}du,$$

but it is awkward when thinking in terms of area (what does it mean to go from 1 to 0 in integral limit?). A more important point is that you will see this reversal of limits will not be available for us in multi-dimension, only the interpretation of area remains. So we need a more consistent way to explain this negative sign.

Following the arguments before, we have

$$du = 2xdx.$$

But if we interpret du, dx as length then the above should be written as

$$du = |2x|dx,$$

or generally

$$du = |f'(x)|dx,$$

if $u = f(x)$. Now when substituing in the integral $\int_{-1}^{0} 2xe^{-x^2}dx$ we see that on $[-1, 0], x$ is negative. Thus the equation $du = |2x|dx$ becomes $du = -(2x)dx$ to preserve the positivity of du. This explains the negative in front of the RHS of

$$\int_{-1}^{0} 2xe^{-x^2}dx = -\int_0^1 e^{-u}du.$$

1.4 Two ways of making substitution

Now that we concluded that the substitution map is invertible, we have two ways to write our change of variables. We can either choose $u = \psi(x)$ or $x = \phi(u)$. The relation between ϕ and ψ is that they are inverses of each other. The difference between the two forms is in how we write the integrand, see more below. We comment that in practice we usually use the substitution $u = \psi(x)$, as the examples given above ($u = x^2$).

Consider the integral $\int_a^b f(x)dx$. It is more convenient here to use the substitution as $x = \phi(u)$. Then the integrand becomes

$$f(x)dx = f(\phi(u))|\phi'(u)|du$$

from the fact that $dx = |\phi'(u)|du$. Suppose in addition that ϕ is an *increasing function* and we make the substitution $x = \phi(u)$. Then the interval $[a, b]$ in x corresponds to the interval $[\phi^{-1}(a), \phi^{-1}(b)] = [\psi(a), \psi(b)]$ in u. The change of variable is

$$\int_a^b f(x)dx = \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(u))|\phi'(u)|du = \int_{\psi(a)}^{\psi(b)} f(\phi(u))|\phi'(u)|du.$$

Note that if ϕ is decreasing then the interval $[a, b]$ in x corresponds to $[\psi(b), \psi(a)]$ in u.

On the other hand, consider the integral $\int_a^b f(\psi(x))\psi'(x)dx$. It is more convenient here to make the substitution $u = \psi(x)$. The integrand becomes

$$f(\psi(x))\psi'(x)dx = \pm f(u)du,$$

where the plus or minus sign depends on the correspondence between $\psi'(x)$ and $|\psi(x)|$ on the interval $[a, b]$, coming from the fact that $du = |\psi'(x)|dx$.

Suppose in addition that ϕ is an *increasing function* and the $|\psi'(x)| = \psi'(x)$ on $[a, b]$. Then the change of variable is

$$\int_a^b f(\psi(x))\psi'(x)dx = \int_{\psi(a)}^{\psi(b)} f(u)du.$$

2 Change of variable in 2-d

2.1 An example

Let the region $\mathcal{R}$ be the parallelogram that connects the four points $(0, 0), (1, 1), (-1, 2), (-2, 1)$. Consider the double integral over $\mathcal{R}$:

$$\iint_{\mathcal{R}} dxdy.$$

This simply calculates the area of the parallelogram $\mathcal{R}$, which can be done by geometric method. Our goal here is to make a change of varaible that maps the region $\mathcal{R}$ to a rectangular region $\mathcal{R}'$ to be determined below.

Indeed consider the map

$$\begin{aligned} x(u,v) &= u-2v \\ y(u,v) &= u+v, \end{aligned}$$

where $(u,v) \in [0,1] \times [0,1]$. If we simply write

$$(x,y) = G(u,v) = (u-2v, u+v)$$

as a map $G : \mathbb{R}^2 \to \mathbb{R}^2$ then we have

$$\begin{aligned} G(0,0) &= (0,0) \\ G(0,1) &= (-2,1) \\ G(1,0) &= (1,1) \\ G(1,1) &= (-1,2) \end{aligned}$$

You can also verify that any point that is inside $\mathcal{R}' = [0,1] \times [0,1]$ is mapped into the parellogram $\mathcal{R}$. For example $G(1/2,1/2) = (-1/2,1)$ is inside $\mathcal{R}$. However, since

$$\iint_{[0,1]\times[0,1]} dudv = 1,$$

clearly

$$\iint_{[0,1]\times[0,1]} dudv \neq \iint_{\mathcal{R}} dxdy.$$

What we are missing is the link between $dudv$ and $dxdy$. As you can guess, the missing link is the ratio between the area of $\mathcal{R}$ and the area of $\mathcal{R}'$.

We will represent the mapping G as a matrix multiplication as followed:

$$G(u,v) = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} := A \begin{bmatrix} u \\ v \end{bmatrix},$$

From our discussion of cross product, you can verify that the area of the paralleogram $\mathcal{R}$ is 3, which is also the absolute value of the determinant of $A : |1\times 1 - (-2)\times 1| = 3$. Thus the correct relation is

$$\iint_{[0,1]\times[0,1]} |\det A| dudv = \iint_{\mathcal{R}} dxdy,$$

where $\mathcal{R}$ is the region arising from the map $G(u,v) = A \begin{bmatrix} u \\ v \end{bmatrix}$ for $(u,v) \in [0,1]\times[0,1]$.

2.2 Change of variables for linear map

The above example motivates us to study the double integral

$$\iint_{[0,1]\times[0,1]} f(G(u,v))|\det A|dudv,$$

where $G(u,v)$ is the linear map

$$G(u,v) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} := A \begin{bmatrix} u \\ v \end{bmatrix}.$$

Clearly we want to make the change of variable $(x,y) = G(u,v)$. We can also verify that the rectangle $[0,1]\times[0,1]$ gets mapped to the parrellogram connecting the four points $(0,0),(a,b),(c,d),(a+c),(b+d)$. The last question is to determine the transformation of $dudv$ under G.

We emphasize here that while $dudv$ is the area of a rectangle of lengths du, dv the image of this rectangle under the map G is NOT a rectangle. Indeed it is a parallelogram that connects the four points $(u,v),(u+du,v),(u,v+dv),(u+du,v+dv)$. We denote the area of this parallelogram as $dG(u,v)$. What we think of as $dxdy$ in the change of variable is actually this $dG(u,v)$:

$$dxdy = dG(u,v) = |\det A|dudv.$$

A picture can also make it clear. Suppose in the $uv-$plane, we partition the rectangle $[0,1]\times[0,1]$ into $10\times 10 = 100$ sub-rectangles, each with lengh and wide 0.1. Then you can verify that the corresponding transformation of these rectangles in the $xy-$ plane is the partition of the parallelogram $(0,0),(a,b),(c,d),(a+c),(b+d)$ into 100 sub-parllelograms. As (u,v) runs over all the grid points on the sub-rectangles of $[0,1]\times[0,1]$, $G(u,v)$ runs over the grid points of the sub-parallelograms of $(0,0),(a,b),(c,d),(a+c),(b+d)$. Thus adding the function $f\circ G$ over the grid points (u,v) multiply by the area of the sub-rectangles times $\det(A)$ is the same as adding the function f over the grid points of the paralleogram $G(u,v)$ multiply by the area of the sub-parallograms.

We conclude by writing out

$$\iint_{[0,1]\times[0,1]} f(G(u,v))|\det A|dudv = \iint_{\mathcal{R}} f(x,y)dxdy.$$

2.3 Change of variables for non-linear map

Consider the integral

$$\iint_{[0,1]\times[0,1]} f(G(u,v))|\det A(u,v)|dudv, \tag{1}$$

where $G(u,v)$ is the non-linear map

$$G(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix},$$

and $|\det A(u,v)|$ is a function in (u,v) to be determined.

Again suppose in the $uv-$plane, we partition the rectangle $[0,1]\times[0,1]$ into $10 \times 10 = 100$ sub-rectangles, each with lengh and wide 0.1. However, note that here the corresponding transformation of these rectangles in the $xy-$ plane will no longer be uniform geometrical shapes. We will still have 100 non-overlapping sub-regions on the xy-plane that partitions the region $\mathcal{R}$ that is the image of $[0,1]\times[0,1]$ under G. Our goal now is to *approximate* the area of each of these sub-regions.

By Taylor first order approximation, we have

$$\begin{bmatrix} x(u+du, v+dv) \\ y(u+du, v+dv) \end{bmatrix} \approx \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial u}x(u,v) & \frac{\partial}{\partial v}x(u,v) \\ \frac{\partial}{\partial u}y(u,v) & \frac{\partial}{\partial v}y(u,v) \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}.$$

We call the matrix $\begin{bmatrix} \frac{\partial}{\partial u}x(u,v) & \frac{\partial}{\partial v}x(u,v) \\ \frac{\partial}{\partial u}y(u,v) & \frac{\partial}{\partial v}y(u,v) \end{bmatrix}$ the Jacobian matrix of (x,y) with respect to (u,v). It is also denoted as $\text{Jac}(G)$ or $\frac{\partial(x,y)}{\partial(u,v)}$. For short hand we just write the absolute value of the determinant of the Jacobian of G as $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

The Taylor approximation tells us that we can approximate the image of the differential rectangle $(u,v), (u+du,v), (u,v+dv), (u+du,v+dv)$ under the map $G(u,v)$ with its image under the linear map $\text{Jac}(G)(u,v)\begin{bmatrix} u \\ v \end{bmatrix}$, which is a parellogram with area $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. That is

$$dG(u,v) \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

This approximation becomes precise in the limit. Thus we see that the function $|\det A(u,v)|$ in (1) is $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. The change of variables formula becomes

$$\iint_{[0,1]\times[0,1]} f(G(u,v))|\det A(u,v)|dudv = \int_{\mathcal{R}} f(x,y)dxdy.$$

Remark: The above analysis also shows that we should write the change of variable as

$$\iint_{[0,1]\times[0,1]} f(G(u,v))dG(u,v),$$

where $dG(u,v)$ is understood as the differential area of the parallelogram on the $xy-$plane that is the image of the differential rectangle on the $uv-$ plane under G. We also found out that $dG(u,v) = |\det \frac{\partial(x,y)}{\partial(u,v)}|dudv$ if G is a linear map. Only in the limiting behavior that we have

$$\iint_{[0,1]\times[0,1]} f(G(u,v))dG(u,v) = \iint_{\mathcal{R}} f(x,y)dxdy.$$